Homework 3: Functions & Linear maps

Deadline: 13th November, 2022

Exercise 1. (3+3+4=10 Points) We define the following four functions:

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + x_2 \\ x_1 x_2 \end{pmatrix},$$

$$f_2 : \mathbb{R} \longrightarrow \mathbb{R} \qquad x \longmapsto \frac{2x}{x^2+4},$$

$$f_3 : \mathbb{R} \longrightarrow \mathbb{R}^2 \qquad x \longmapsto \begin{pmatrix} 3\cos(x) \\ 2\sin(x) \end{pmatrix},$$

$$f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 - x_2 \\ x_1 - 3x_2 \\ x_1 - x_2 \end{pmatrix}.$$

i) Calculate the image of each function, i.e. describe $\mathrm{im}(f_j)$ for $j = 1, 2, 3, 4$ as explicit as possible. If you can not find a mathematical description try to describe the elements of the image in words.

ii) Decide for each function if it is injective and/or surjective and/or bijective.

iii) Decide which of the above functions are linear maps.

Justify your answers in ii) and iii).

Exercise 2. (5 Points) Show that there exist a unique linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$G\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad G\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

What is the value of $G(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of G.

Exercise 3. (2+2+3=7 Points)

i) Let X be a finite set. Show that a function $f : X \to X$ is injective if and only if it is surjective.

ii) Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map. Show that F can not be injective.

iii) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that F is injective if and only if the only solution to $F(x) = 0$ is $x = 0$.

For Exercise 1i) see videos "Image of a non-linear map" and "When is a linear map surjective/invective" on the homepage.

1) (10 Points) Consider the following linear system

$$\left\{\begin{array}{ccccccccc} 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & = & 6 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 5 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & = & 7 \end{array}\right. .$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A.

iii) Find all the solutions to the linear system.

iv) Calculate the rank of $(A \mid b)$ and A.

v) Find a vector $c \in \mathbb{R}^3$, such that $Ax = c$ has no solutions. Calculate the rank of $(A \mid c)$.

2) (10 Points) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$\begin{aligned} f_1 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto (u \bullet x)u + x\,, \end{aligned} \qquad \begin{aligned} f_2 : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \begin{pmatrix} 2\cos(x) \\ \sin(x) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} f_3 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \frac{x \bullet x}{u \bullet u} u\,, \end{aligned} \qquad \begin{aligned} f_4 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^4 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\longmapsto \begin{pmatrix} 0 \\ x_1 + 2x_2 + 3x_3 \\ x_1 - x_3 \\ x_2 \end{pmatrix}. \end{aligned}$$

i) Which of the above functions f_1, f_2, f_3, f_4 are linear maps? For each one that is linear, determine its matrix.

ii) Draw a picture of the image of f_2. Is f_2 injective and/or surjective?

3) (6 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad G\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

i) Determine the matrix of G.

ii) Determine the matrix of $G \circ G$.

4) (6 Points) We define the following linear map

$$\begin{aligned} H : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 \\ x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}. \end{aligned}$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find all vectors $v \in \mathbb{R}^3$, which are orthogonal to all vectors in the image of H.

1) (10 Points) Consider the following linear system

$$\begin{cases} -2x_1 + 4x_2 + x_3 + x_4 = 6 \\ -3x_1 + 6x_2 + x_3 = 7 \\ x_1 - 2x_2 + x_4 = -1 \end{cases}.$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A.

iii) Find all the solutions to the linear system.

iv) Calculate the rank of $(A \mid b)$ and A.

v) Find all $y \in \mathbb{R}^4$ with $Ay = 2b$ by using your result for iii).

2) (8 Points) Let $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad x \longmapsto \begin{pmatrix} u \bullet x \\ 0 \\ x \bullet u \end{pmatrix}, \qquad f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto 2^{x_1+x_2} - 1, \qquad f_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + x_2 x_3 \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad G\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$$F\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Show that F is not a linear map.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \\ x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find a non-zero vector $v \in \mathbb{R}$, such that v is orthogonal to $H(v)$. (Just one explicit vector is enough)

After finishing this exam please send your solution as one pdf file to
henrik.bachmann@math.nagoya-u.ac.jp

1) (10 Points) Consider the following linear system

$$\begin{cases} 3x_1 - 6x_2 + x_3 + 5x_4 = 5 \\ 2x_1 - 4x_2 + x_3 + 3x_4 = 4 \\ -x_1 + 2x_2 - 2x_3 = -5 \end{cases} .$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

iii) Find all the solutions to the linear system.

iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which are orthogonal to the vector $u = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \sin(x_1) + \cos(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad x \longmapsto \begin{pmatrix} x \bullet x \\ 0 \\ u \bullet u \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with

$$F\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}.$$

Show that F is not injective.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + 2x_2 \\ x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find all vectors $x \in \mathbb{R}^3$ with $H(x) = 2x$.

After finishing this exam submit your solution as one pdf file at NUCT at the "Midterm" assignment.