Tutorial 13: Orthonormal basis & Gram-Schmidt

A basis $F = (f_1, \dots, f_m)$ of a subspace U is called an **orthonormal basis (ONB)** of U if $f_1, \dots, f_m$ are orthonormal. In other words, this means that:

(i) The vectors are pairwise orthogonal: For $1 \leq i \neq j \leq m$ we have $f_i \bullet f_j = 0$.

(ii) The vectors are normalized to norm 1: For all $1 \leq i \leq m$ we have $\|f_i\| = \sqrt{f_i \bullet f_i} = 1$.

Gram-Schmidt algorithm (GSA)

Let $B = (b_1, \dots, b_m)$ be an arbitrary basis of a subspace $U \subset \mathbb{R}^n$. The GSA constructs an orthonormal basis $F = (f_1, \dots, f_m)$ of U out of the basis B in the following m steps:

Step 1: Set $f_1 = \widehat{b_1} = \frac{1}{\|b_1\|} b_1$.

Step l ($2 \leq l \leq m$): We have constructed orthonormal vectors $f_1, \dots, f_{l-1}$ in the steps before. Now set

$$w_l = b_l - (b_l \bullet f_1) f_1 - \cdots - (b_l \bullet f_{l-1}) f_{l-1} = b_l - \sum_{i=1}^{l-1} (b_l \bullet f_i) f_i$$

and define $f_l = \frac{1}{\|w_l\|} w_l$.

Exercise 1. (Continue from the lecture) Consider the basis $B = (b_1, b_2, b_3)$ of $\mathbb{R}^3$, where

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad b_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

(i) Use the Gram-Schmidt algorithm to construct an orthonormal basis F of $\mathbb{R}^3$ out of B.

(ii) Find orthonormal bases for the subspaces $U = \text{span}\{b_1, b_2\}$ and $V = \text{span}\{b_1, b_3\}$.

See `https://tinyurl.com/yc5mjem6` for a visualization.

1) (14 Points) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ -1 & 2 & 5 \end{pmatrix}$.

i) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.

ii) Calculate the matrix BA and decide if BA^n is invertible for any integer $n \geq 1$.

iii) Determine if $\mathrm{im}(A) \cup \mathrm{im}(B)$ and $\mathrm{im}(A) \cap \mathrm{im}(B)$ are subspaces and, if they are, determine a basis.

2) (12 Points) We define the subspace $U = \mathrm{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$ and $[u_4]_B$, where B is the basis from i).

iii) Calculate an orthonormal basis $F = (f_1, \dots, f_m)$ for U and determine $[u_1]_F$, $[u_2]_F$,$[u_3]_F$ and $[u_4]_F$

iv) Determine a basis for $U^\perp$.

3) (12 Points) Set $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and let $C \in \mathbb{R}^{2\times 2}$ be an arbitrary matrix. Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{x \in \mathbb{R}^2 \mid x \bullet x = x \bullet u\right\}$.

ii) $U_2 = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 - x_2 = 3x_2\right\}$.

iii) $U_3 = \left\{x \in \mathbb{R}^2 \mid Cx = x\right\}$.

iv) $U_4 = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \cdot x_2 \geq x_1\right\}$.

4) (12 Points) Assume we have the following data points

i	1	2	3
x_i	1	2	3
y_i	0	-1	-3

i) Find the line of best fit for the above data, i.e. find $m, n \in \mathbb{R}$ such that the function $l(x) = mx + n$ minimizes the sum of squares $\sum_{i=1}^{3} (l(x_i) - y_i)^2$.

ii) We define the following linear map

$$H : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and set $V = \mathrm{im}(H)$. Determine a basis of V and for $b = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$ calculate the orthogonal projection $P_V(b)$. Use your result to show that b is not an element in V.

After finishing this exam submit your solution as one pdf file at NUCT at the "Final exam" assignment.