Tutorial 12: Coordinates

Happy new year! あけましたおめでとうございます! Frohes neues Jahr!

Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $V \subset \mathbb{R}^n$. We define the **coordinate map** by

$$\begin{aligned} c_B : \mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} &\longmapsto \lambda_1 b_1 + \dots + \lambda_m b_m \,. \end{aligned}$$

i) Since $b_1, \dots, b_m$ are linearly independent the map c_B is injective.

ii) Since $b_1, \dots, b_m$ span V, i.e. $V = \operatorname{span}\{b_1, \dots, b_m\}$, we have $\operatorname{im}(c_B) = V$.

iii) From i) and ii) we get: The map $c_B : \mathbb{R}^m \longrightarrow V$ is bijective.

iv) For all $x \in V$ there exist unique $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$x = \lambda_1 b_1 + \dots + \lambda_m b_m \,.$$

The numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ are the **coordinates of x (in the basis B)**.

v) The **coordinate vector** of x (with respect to B) is given by

$$[x]_B = c_B^{-1}(x) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} .$$

Exercise 1. Consider the subspace $V = \operatorname{span}\{v_1, v_2\}$ of $\mathbb{R}^3$ where

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

(i) Find two different bases B_1 and B_2 of V.

(ii) For the element $x = \begin{pmatrix} 5 \\ -2 \\ -8 \end{pmatrix}$ decide if x is in V. If this is the case, calculate the coordinate vectors $[x]_{B_1}$ and $[x]_{B_2}$.

Let $B = (b_1, \dots, b_n)$ be a basis of $\mathbb{R}^n$. The **change-of-basis matrix** associated with B is

$$S_B = [c_B] = \begin{pmatrix} | & & | \\ b_1 & \dots & b_n \\ | & & | \end{pmatrix} .$$

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, B_1 be a basis of $\mathbb{R}^n$ and B_2 be a basis of $\mathbb{R}^m$. The **matrix of** F **with respect to** B_1 **and** B_2 is the matrix

$$[F]_{B_1}^{B_2} := [c_{B_2}^{-1} \circ F \circ c_{B_1}] = S_{B_2}^{-1}[F]S_{B_1} .$$

In the case $n = m$ and $B_1 = B_2$ we just write $[F]_{B_1} := [F]_{B_1}^{B_1}$. (This will often be the case)

Motivation: The properties of a linear map can often be easier understand when writing the matrix of F in a basis different to the standard basis. Also: This helps to decide if two linear maps are "essentially" the same.

Exercise 2. Let $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(i) Determine the matrices $[P_u]$ and $[P_v]$. Can you determine $[P_u]$ without doing any calculation?

(ii) Would you say that P_u and P_v are really "different" linear maps?

(iii) Find a basis B of $\mathbb{R}^2$, such that $[P_v]_B = [P_u]$.