

1) (10 Points) Consider the following linear system

$$\begin{cases} x_1 + 3x_2 + x_4 = 1 \\ x_2 + 2x_3 - 2x_4 = 2 \\ 2x_1 - 2x_2 + x_3 + x_4 = 3 \end{cases} .$$

- i) Find a matrix $A \in \mathbb{R}^{3 \times 4}$ and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.
- ii) Determine the row-reduced echelon forms of the matrices $(A | b)$ and A and calculate their ranks.
- iii) Find all the solutions to the linear system.
- iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which have norm $\|x\| = \sqrt{14}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} (u \bullet u) - 2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto e^{x_1} - e^{x_2}, \qquad x \mapsto (x \bullet u)u .$$

- i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.
- ii) Is f_2 injective and/or surjective?

3) (8 Points) Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map with

$$G \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad G \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} .$$

- i) Determine the matrix of G .
- ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to $G(x)$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ 2x_1 - 2x_3 \end{pmatrix} .$$

- i) Calculate the image of H .
- ii) Decide if H is injective and/or surjective.
- iii) Find a linear map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ with $\text{im}(F) = \text{im}(H)$.