1) (12 Points) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -4 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ and $C = A^T B A$.

i) Determine whether or not the matrices A, B, and C are invertible and, if they are, compute their inverses.

ii) Determine $\operatorname{im}(C)$, $\ker(C)$ and $\operatorname{im}(C) \cap \ker(C)$.

iii) Give a basis for $\ker(C^n)$ for all $n \geq 1$.

2) (14 Points) We define the subspace $U = \operatorname{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -6 \\ -6 \\ -3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$ and $[u_4]_B$, where B is the basis from i).

iii) Find a linear map $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $\ker(F) = U^\perp$ and determine a basis of $\operatorname{im}(F)$.

3) (12 Points) Let $D \in \mathbb{R}^{2\times 2}$ be an arbitrary matrix. Which of the following sets are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid 2x_1 - x_2 = -x_3 + x_2 \right\}$.

ii) $U_2 = \left\{ x \in \mathbb{R}^{2023} \mid x \bullet x \geq -2023 \right\}$.

iii) $U_3 = \left\{ x \in \mathbb{R}^2 \mid \text{ There exists a } y \in \mathbb{R}^2 \text{ with } Dy = 3x \right\}$.

iv) $U_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\} \cup \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \cdot x_2 \leq 0 \right\}$.

4) (12 Points) We consider the vector $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, the linear map

$$G : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and define the subspace $U = \operatorname{im}(G)$.

i) Show that $\dim(U) = 2$ and find an orthonormal basis $F = (f_1, f_2)$ of U.

ii) Determine the orthogonal projection $y = P_U(b)$ of b onto U and calculate $[y]_F$.

iii) Find a $x \in \mathbb{R}^2$ such that $\|G(x) - b\|$ is minimal.