Linear Algebra I & Mathematics Tutorial 1b, menti.com: 2285 8160

Tutorial 12, January 11th, 2022, 14:45 - 15:28

Vectors $u_1, \dots, u_l \in \mathbb{R}^n$ *are called* **orthonormal** *if for* $1 \leq i, j \leq l$

$$u_i \bullet u_j = \begin{cases} 1 & , \text{if } i = j \\ 0 & , \text{if } i \neq j \end{cases} .$$

Today:

- Lecture 12
- Final exam 19/20

Tutorial Exercise:

Let $b_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Find an orthonormal basis $F = (f_1, f_2)$ of $V = \operatorname{span}\{b_1, b_2\}$.

Definition 12.2. *A basis* $B = (b_1, \dots, b_m)$ *of a subspace* U *is called an* **orthonormal basis (ONB)** *of* U *if* $b_1, \dots, b_m$ *are orthonormal.*

Definition 12.4. *For a subspace* $U \subset \mathbb{R}^n$ *we define the* **orthogonal complement of** U **in** $\mathbb{R}^n$ *by*

$$U^{\perp} = \{x \in \mathbb{R}^n \mid x \bullet u = 0 \text{ for all } u \in U\} .$$

10 Bases & dimensions

Definition 10.1. *Let $V \subset \mathbb{R}^n$ be a subspace. Vectors $v_1, \dots, v_l \in V$ form a* **basis of V** *if*

i) $V = \operatorname{span}\{v_1, \dots, v_l\}$,

ii) $v_1, \dots, v_l$ *are linearly independent.*

In this case we also say that $\{v_1, \dots, v_l\}$ is a basis of V.

Later we will also be interested in the order of the v_j and write a basis as a tuple $(v_1, \dots, v_l)$.

Theorem 10.2. *For any subspace $V \subset \mathbb{R}^n$ we have the following:*

i) V has a basis.

ii) All bases of V have the same number of elements.

iii) If $v_1, \dots, v_l \in V$ are linearly independent then there exist $u_{l+1}, \dots, u_t \in V$, such that $\{v_1, \dots, v_l, u_{l+1}, \dots, u_t\}$ is a basis of V.

iv) If $V = \operatorname{span}\{w_1, \dots, w_m\}$ then there exists a subset $\{u_1, \dots, u_t\} \subset \{w_1, \dots, w_m\}$ such that $\{u_1, \dots, u_t\}$ is a basis of V.

Definition 10.3. *Let $V \subset \mathbb{R}^n$ be a subspace. The* **dimension of V**, *denoted by* $\dim(V)$, *is the number of elements in a basis of V.*

Corollary 10.4. *Let $V \subset \mathbb{R}^n$ be a subspace with $\dim(V) = m$ and $v_1, \dots, v_m \in V$. Then the following statements are equivalent:*

i) $v_1, \dots, v_m$ are linearly independent.

ii) $V = \operatorname{span}\{v_1, \dots, v_m\}$.

iii) $\{v_1, \dots, v_m\}$ is a basis of V.

Theorem 10.5. *For a linear map $F : \mathbb{R}^n \to \mathbb{R}^m$ we have*

$$n = \dim(\ker(F)) + \dim(\operatorname{im}(F))\,.$$

11 Coordinates

From now on we will consider ordered bases, which means that we will write $(b_1, \dots, b_n)$ (a tuple) for a basis instead of $\{b_1, \dots, b_n\}$ (a set). The difference is, that we care about the order now. For example, the two sets $\{b_1, b_2\} = \{b_2, b_1\}$ are the same, but $(b_1, b_2) \neq (b_2, b_1)$.

Definition 11.1. *Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $V \subset \mathbb{R}^n$. We define the* **coordinate map** *by*

$$\begin{aligned} c_B : \mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} &\longmapsto \lambda_1 b_1 + \cdots + \lambda_m b_m\,. \end{aligned}$$

Theorem 11.2. *Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $V \subset \mathbb{R}^n$.*

i) The coordinate map $c_B : \mathbb{R}^m \longrightarrow V$ is bijective.

ii) For all $x \in V$ there exist unique $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$x = \lambda_1 b_1 + \cdots + \lambda_m b_m \,.$$

Definition 11.3. *Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $V \subset \mathbb{R}^n$ and $x \in V$ with*

$$x = \lambda_1 b_1 + \cdots + \lambda_m b_m \,.$$

i) The numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ are the **coordinates of x (in the basis B)**.

ii) The **coordinate vector** *of x (with respect to B) is given by*

$$[x]_B = c_B^{-1}(x) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} .$$

Proposition 11.4. *Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $V \subset \mathbb{R}^n$, $x, y \in V$ and $\mu \in \mathbb{R}$. Then*

i) $[x + y]_B = [x]_B + [y]_B$,

ii) $[\mu x]_B = \mu [x]_B$,

iii) $[0]_B = 0$.

Definition 11.5. *Let $B = (b_1, \dots, b_n)$ be a basis of $\mathbb{R}^n$. The* **change-of-basis matrix** *associated with B is*

$$S_B = [c_B] = \begin{pmatrix} | & & | \\ b_1 & \dots & b_n \\ | & & | \end{pmatrix} .$$

Definition 11.6. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, B_1 be a basis of $\mathbb{R}^n$ and B_2 be a basis of $\mathbb{R}^m$. The* **matrix of F with respect to B_1 and B_2** *is the matrix*

$$[F]_{B_1}^{B_2} := [c_{B_2}^{-1} \circ F \circ c_{B_1}] \,.$$

In the case $n = m$ and $B_1 = B_2$ we just write $[F]_{B_1} := [F]_{B_1}^{B_1}$.

Proposition 11.7. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, B_1 be a basis of $\mathbb{R}^n$ and B_2 be a basis of $\mathbb{R}^m$.*

i) We have

$$[F]_{B_1}^{B_2} = S_{B_2}^{-1} [F] S_{B_1} \,.$$

ii) If $B_1 = (b_1, \dots, b_n)$ then

$$[F]_{B_1}^{B_2} = \begin{pmatrix} | & & | \\ [F(b_1)]_{B_2} & \dots & [F(b_n)]_{B_2} \\ | & & | \end{pmatrix} .$$

12 Orthonormal bases & Gram-Schmidt algorithm

Definition 12.1. *i) A vector $u \in \mathbb{R}^n$ is called a* **unit vector** *if $\|u\| = 1$. (i.e. $u \bullet u = 1$)*

ii) Every vector $u \in \mathbb{R}^n$ with $u \neq 0$ can be normalized by

$$\widehat{u} = \frac{1}{\|u\|} u \,.$$

The vector $\hat{u}$ is a unit vector and shows in the same direction as u.

iii) Vectors $u_1, \ldots, u_l \in \mathbb{R}^n$ are called **orthonormal** *if for $1 \leq i, j \leq l$*

$$u_i \bullet u_j = \begin{cases} 1 & , \text{if } i = j \\ 0 & , \text{if } i \neq j \end{cases} .$$

Definition 12.2. *A basis $B = (b_1, \ldots, b_m)$ of a subspace U is called an* **orthonormal basis (ONB)** *of U if $b_1, \ldots, b_m$ are orthonormal.*

Proposition 12.3. *i) If $v_1, \ldots, v_m \in \mathbb{R}^n$ are orthonormal (ON), then they are linearly independent.*

ii) Let $B = (v_1, \ldots, v_m)$ be an ONB of $V \subset \mathbb{R}^n$ and $u \in V$. Then

$$[u]_B = \begin{pmatrix} u \bullet v_1 \\ \vdots \\ u \bullet v_m \end{pmatrix} \in \mathbb{R}^m \,,$$

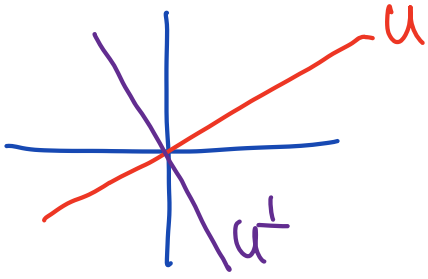
i.e. $u = \sum_{i=1}^{m} (u \bullet v_i) v_i$.

iii) If $B = (v_1, \ldots, v_m)$ is an ONB of $V \subset \mathbb{R}^n$ and $u, w \in V$, then

$$u \bullet w = [u]_B \bullet [w]_B \,.$$

Definition 12.4. *For a subspace $U \subset \mathbb{R}^n$ we define the* **orthogonal complement of U in $\mathbb{R}^n$** *by*

$$U^\perp = \{x \in \mathbb{R}^n \mid x \bullet u = 0 \text{ for all } u \in U\} \,.$$



Lemma 12.5. *Let $U \subset \mathbb{R}^n$ be a subspace.*

i) $U^\perp \subset \mathbb{R}^n$ is a subspace.

ii) We have $U \cap U^\perp = \{0\}$.

iii) If $(u_1, \ldots, u_r)$ is a basis of U, $x \in \mathbb{R}^n$, then

$$x \in U^\perp \quad \Longleftrightarrow \quad x \bullet u_1 = \cdots = x \bullet u_r = 0 \,.$$

iv) Let $(f_1, \dots, f_r)$ be an ONB of U and $x \in \mathbb{R}^n$. Then

$$x = x_\parallel + x_\perp \,,$$

where

$$x_\parallel = \sum_{i=1}^{r} (x \bullet f_i) f_i \in U$$
$$x_\perp = x - x_\parallel \in U^\perp \,.$$

Gram-Schmidt algorithm (GSA)

Let $B = (b_1, \dots, b_m)$ be an arbitrary basis of a subspace $U \subset \mathbb{R}^n$. The GSA constructs an orthonormal basis $F = (f_1, \dots, f_m)$ of U out of the basis B in the following way m steps:

Step 1**:** Set $f_1 = \widehat{b_1} = \frac{1}{\|b_1\|} b_1$.

Step l $(2 \leq l \leq m)$: We have constructed orthonormal vectors $f_1, \dots, f_{l-1}$ in the steps before. Now set

$$w_l = b_l - (b_l \bullet f_1) f_1 - \cdots - (b_l \bullet f_{l-1}) f_{l-1} = b_l - \sum_{i=1}^{l-1} (b_l \bullet f_i) f_i$$

and define $f_l = \frac{1}{\|w_l\|} w_l$.

Theorem 12.6. *Every subspace of $\mathbb{R}^n$ has an ONB.*

Corollary 12.7. *Let $U \subset \mathbb{R}^n$ be a subspace. For all $x \in \mathbb{R}^n$ there exist unique $x_\parallel \in U$ and $x_\perp \in U^\perp$ with*

$$x = x_\parallel + x_\perp \,.$$

13 Orthogonal Projection & Least squares

Definition 13.1. *Let $U \subset \mathbb{R}^n$ be a subspace. The map*

$$P_U : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longmapsto x_\parallel$$

is the **orthogonal projection** *onto U.*

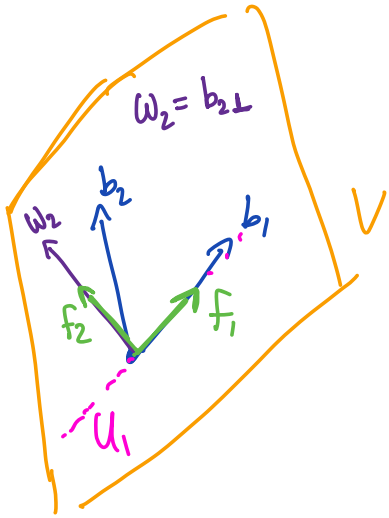
Proposition 13.2. *Let $U \subset \mathbb{R}^n$ be a subspace.*

i) P_U is a linear map.

Tutorial Exercise:

Let $b_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Find an orthonormal basis $F = (f_1, f_2)$ of $V = \text{span}\{b_1, b_2\}$.



iv) Let $(f_1, \ldots, f_r)$ be an ONB of U and $x \in \mathbb{R}^n$. Then

$$x = x_\parallel + x_\perp,$$

where

$$x_\parallel = \sum_{i=1}^{r} (x \bullet f_i) f_i \in U$$
$$x_\perp = x - x_\parallel \in U^\perp.$$

Step 1: Set $f_1 = \hat{b}_1 = \frac{1}{\|b_1\|} b_1$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

$$U_1 = \text{span}\{f_1\} = \text{span}\{b_1\}$$

Step 2.1: $w_2 = b_{2\perp} = b_2 - b_{2\parallel}$

$$= b_2 - \sum_{i=1}^{1} (b_2 \bullet f_i) f_i$$

$$= b_2 - (b_2 \bullet f_1) f_1$$

$$= \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \frac{3}{\sqrt{5}} \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \frac{3}{5} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{8}{5} \\ 2 \\ \frac{4}{5} \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 \\ 10 \\ 4 \end{pmatrix} = \frac{2}{5} \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$$

$$b_2 \bullet f_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} 3$$

(Check: $w_2 \bullet f_1 = 0$)

Step 2.2 Set $f_2 = \hat{w}_2 = \frac{1}{\sqrt{4^2 + 5^2 + 2^2}} \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix} = \frac{1}{3\sqrt{5}} \begin{pmatrix} 4 \\ 5 \\ 2 \end{pmatrix}$.

$16 + 25 + 4 = 45 = 9 \cdot 5$

$$U_2 = \operatorname{span}\{f_1, f_2\} = \operatorname{span}\{b_1, f_2\}$$
$$= \operatorname{span}\{b_1, b_2\} = V.$$

1) (12 Points) Let $A = \begin{pmatrix} 0 & 1 & -2 & 3 \\ 1 & -2 & 3 & -4 \\ -2 & 3 & -4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

i) Compute the products AB and BA, or explain why they are not defined.

ii) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.

iii) Calculate $\mathrm{im}(B)$ and $\ker(B)$.

2) (14 Points) We define the subspace $U = \mathrm{span}\{u_1, u_2, u_3\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \ldots, b_m)$ of U and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$ and $[u_3]_B$, where B is the basis you determined in i).

iii) Determine a basis for $U^\perp$.

iv) Find a linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with $\ker(G) = \{0\}$ and $\mathrm{im}(G) = U$.

3) (10 Points) Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = x_1 x_2 \right\}$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 = x_1 + x_2 \right\}$.

iii) $U_3 = \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \cup \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

(Friendly reminder: $\cup$ is the union of two sets)

4) (14 Points) We define the following linear map

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

i) Calculate an orthonormal basis $F = (f_1, \ldots, f_r)$ for $\mathrm{im}(T)$.

ii) Check for which $t \in \mathbb{R}$ the vector $v = \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}$ is an element in $\mathrm{im}(T)$. Determine the coordinate vector $[v]_F$ in this case.

iii) Find a $w \in \mathbb{R}^3$ with $[w]_F = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

iv) Find a $x \in \mathbb{R}^2$ such that $\|T(x) - b\|$ is minimal, where $b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

(In ii) and iii) the F is the basis of $\mathrm{im}(T)$ you calculated in i)).

$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$ ⟵ Solutions to this are:

$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \sim \ldots \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\mathcal{B} = (u_1, u_2)$

$[u_1]_{\mathcal{B}} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$u_1 = \lambda_1 u_1 + \lambda_2 u_2$

$[u_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$[u_3]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_1 = -2t$

$\lambda_2 = -t$

$\lambda_3 = t$

$-2t\,u_1 - t\,u_2 + t\,u_3 = 0$

$t = 1$:

$u_3 = 2u_1 + 1u_2$

$U = \operatorname{span}\{u_1, u_2\}$

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iii) Determine a basis for $U^{\perp}$.

$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix},$

$U^{\perp} = \{ x \in \mathbb{R}^3 \mid x \cdot u = 0 \text{ for all } u \in U \}$

$= \{ x \in \mathbb{R}^3 \mid x \cdot u_1 = x \cdot u_2 = 0 \} = \operatorname{span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}$

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\quad X \cdot u_1 = x_2 + x_3 = 0$

$X \cdot u_2 = x_1 - 3x_3 = 0$

$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solution: $X = t \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ $\quad t \in \mathbb{R}$

1) (12 Points) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

i) Compute the products AB and BA, or explain why they are not defined.

ii) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.

iii) Find all $x \in \mathbb{R}^3$ with $A^TAA^TAA^TAx = 0$. Justify you answer.

2) (12 Points) We define the subspace $U = \text{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$ and $[u_4]_B$, where B is the basis from i).

iii) Find a linear map $G : \mathbb{R}^3 \to \mathbb{R}^3$ with $\text{im}(G) = U$. What is the dimension of $\ker(G)$?

3) (12 Points) Set $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 - 3x_2 = x_1 \right\}$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \text{ is an integer, i.e. } x_1 \in \{\dots, -2, -1, 0, 1, 2, \dots\} \right\}$.

iii) $U_3 = \left\{ x \in \mathbb{R}^2 \mid x \notin \text{span}\{u\} \right\}$.

iv) $U_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet u = x_1 \right\}$.

4) (14 Points) We define the following linear map

$$\begin{aligned} H : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & 1 \\ -2 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

i) Show that $\dim(\text{im}(H)) = 2$.

ii) Calculate an orthonormal basis (f_1, f_2) for $\text{im}(H)$.

iii) Find a vector $v \in \mathbb{R}^3$, such that $B = (f_1, f_2, v)$ is an orthonormal basis for $\mathbb{R}^3$.

iv) Calculate $[H(x)]_B$ for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

v) Find a $x \in \mathbb{R}^2$ such that $\|H(x) - b\|$ is minimal, where $b = \begin{pmatrix} -5 \\ 1 \\ -1 \end{pmatrix}$.

After finishing this exam please send your solution as **one pdf file** to
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