Solution

Linear Algebra I - Midterm exam Fall 2021

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1) (10 Points) Consider the following linear system

$$\left\{\begin{array}{rcrcrcrcr} 3x_1 & - & 6x_2 & + & x_3 & + & 5x_4 & = & 5 \\ 2x_1 & - & 4x_2 & + & x_3 & + & 3x_4 & = & 4 \\ -x_1 & + & 2x_2 & - & 2x_3 & & & = & -5 \end{array}\right. .$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

iii) Find all the solutions to the linear system.

iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which are orthogonal to the vector $u = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \sin(x_1) + \cos(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad x \longmapsto \begin{pmatrix} x \bullet x \\ 0 \\ u \bullet u \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with

$$F\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}.$$

Show that F is not injective.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + 2x_2 \\ x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find all vectors $x \in \mathbb{R}^3$ with $H(x) = 2x$.

After finishing this exam submit your solution as one pdf file at NUCT at the "Midterm" assignment.

1) (10 Points) Consider the following linear system

$$\begin{cases} 3x_1 - 6x_2 + x_3 + 5x_4 = 5 \\ 2x_1 - 4x_2 + x_3 + 3x_4 = 4 \\ -x_1 + 2x_2 - 2x_3 = -5 \end{cases}.$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

iii) Find all the solutions to the linear system.

iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which are orthogonal to the vector $u = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

i) $A = \begin{pmatrix} 3 & -6 & 1 & 5 \\ 2 & -4 & 1 & 3 \\ -1 & 2 & -2 & 0 \end{pmatrix}$ $\quad b = \begin{pmatrix} 5 \\ 4 \\ -5 \end{pmatrix}$

ii) $(A|b) = \left(\begin{array}{cccc|c} 3 & -6 & 1 & 5 & 5 \\ 2 & -4 & 1 & 3 & 4 \\ -1 & 2 & -2 & 0 & -5 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 2 & 0 & 5 \\ 2 & -4 & 1 & 3 & 4 \\ 3 & -6 & 1 & 5 & 5 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 1 & -2 & 2 & 0 & 5 \\ 0 & 0 & -3 & 3 & -6 \\ 0 & 0 & -5 & 5 & -10 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -2 & 2 & 0 & 5 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \mathrm{rref}(A|b)$

$= \mathrm{rref}(A)$.

The ranks are $\mathrm{rk}(A|b) = \mathrm{rk}(A) = 2$.

iii) From the rref$(A|b)$ we can see that the solutions of $Ax=b$ are given by

$$\underline{x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} \text{ with } (\star)\begin{cases} x_1 = 1+2t_1-2t_2 \text{ for } t_1, t_2 \in \mathbb{R} \\ x_2 = t_1 \\ x_3 = 2+t_2 \\ x_4 = t_2 \end{cases}$$

iv) A solution $x \in \mathbb{R}^4$ from iii) is orthogonal to $u=\begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$ if $x \cdot u = 0$.

With x as in $(\star)$ we get

$$x \cdot u = t_1 - (2+t_2) + t_2 = t_1 - 2.$$

Therefore $x \cdot u = 0$ if $t_1 = 2$.

$\Rightarrow$ The vectors $x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ with $x_1 = 5-2t$ for $t \in \mathbb{R}$

$x_2 = 2$

$x_3 = 2+t$

$x_4 = t$

satisfy $Ax=b$ and $x \cdot u = 0$

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \sin(x_1) + \cos(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \qquad x \longmapsto \begin{pmatrix} x \bullet x \\ 0 \\ u \bullet u \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

i) • We have $u \bullet u = 1^2 + 2^2 = 5$, i.e.

$$f_1\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + 5x_3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and therefore f_1 is linear with $[f_1] = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

• f_2 is <u>not</u> linear, because $\sin(0) = 0$, $\cos(0) = 1$ and therefore

$$f_2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \sin(0) + \cos(0) = 1$$

and $f\left(2 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1$.

$$\Rightarrow \underbrace{2 \cdot f\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{2} \neq \underbrace{f\left(2 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}_{1}$$

• We have $f_3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ 0 \\ 5 \end{pmatrix}$, which is <u>not</u> linear since $2 \cdot f_3\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \neq f_3\left(2 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

ii) f_2 is <u>not surjective</u>, since $\sin(x) \leq 1$ and $\cos(x) \leq 1$ and therefore $\sin(x_1) + \cos(x_2) \leq 2$.

$\Rightarrow 3 \notin \operatorname{im}(f_2)$.

f_2 is <u>not injective</u> since

$$f_2\begin{pmatrix}0\\0\end{pmatrix} = f_2\begin{pmatrix}2\pi\\2\pi\end{pmatrix} = 1, \text{ but } \begin{pmatrix}0\\0\end{pmatrix} \neq \begin{pmatrix}2\pi\\2\pi\end{pmatrix}.$$

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad G\begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}-2\\2\end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with

$$F\begin{pmatrix}-1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\2\\3\end{pmatrix}, \quad F\begin{pmatrix}1\\-1\\5\end{pmatrix} = \begin{pmatrix}6\\4\\6\end{pmatrix}.$$

Show that F is not injective.

i) We first try to write $\begin{pmatrix}x_1\\x_2\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}-2\\-1\end{pmatrix}$ for arbitrary $x_1, x_2 \in \mathbb{R}$.

$$\begin{array}{c} -1 \\ \hookrightarrow \end{array}\left(\begin{array}{cc|c} 1 & -2 & x_1 \\ 1 & -1 & x_2 \end{array}\right) \sim \begin{array}{c} \\ 2 \end{array}\left(\begin{array}{cc|c} 1 & -2 & x_1 \\ 0 & 1 & x_2 - x_1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -x_1 + 2x_2 \\ 0 & 1 & x_2 - x_1 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix}x_1\\x_2\end{pmatrix} = (-x_1 + 2x_2)\begin{pmatrix}1\\1\end{pmatrix} + (x_2 - x_1)\begin{pmatrix}-2\\-1\end{pmatrix}.$$

Since G is linear we get

$$G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-x_1+2x_2)\, G\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (x_2-x_1)\, G\begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$= (-x_1+2x_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (x_2-x_1)\begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -x_1+2x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2x_1-2x_2 \\ -2x_1+2x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ -2x_1+2x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}}_{[G]}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

ii) Since F is linear we have $F(\lambda x) = \lambda x$.

In particular for $\lambda = 2$ and $x = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ we get

$$F\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = F\left(2\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right) = 2F\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}.$$

Therefore $F\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = F\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}$, but $\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$.

$\Rightarrow$ F is not injective.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + 2x_2 \\ x_2 + x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{[H]} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find all vectors $x \in \mathbb{R}^3$ with $H(x) = 2x$.

i) We want to find all $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ such that $[H]x = y$ has a solution:

$$([H] \mid y) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & y_1 \\ 1 & 2 & 0 & y_2 \\ 0 & 1 & 1 & y_3 \end{array}\right) \overset{-1}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -1 & y_1 \\ 0 & 1 & 1 & y_2 - y_1 \\ 0 & 1 & 1 & y_3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -2 & 2y_1 - y_2 \\ 0 & 1 & 1 & y_2 - y_1 \\ 0 & 0 & 0 & y_3 + y_1 - y_2 \end{array}\right)$$

$\Rightarrow$ We only get a solution if $y_3 + y_1 - y_2 = 0$.

$\Rightarrow$ $\operatorname{im}(H) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y_1 - y_2 + y_3 = 0 \right\}$.

ii) By i) we see that H is not surjective, since $\operatorname{im}(H) \neq \mathbb{R}^3$. For example $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin \operatorname{im}(H)$.

H is not injective since

$$H\begin{pmatrix}0\\0\\0\end{pmatrix} = H\begin{pmatrix}2\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

but $\begin{pmatrix}0\\0\\0\end{pmatrix} \neq \begin{pmatrix}2\\-1\\1\end{pmatrix}$. (This solution can be obtained from i) by taking $y=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and choosing $t=0$ and $t=1$ for the free variable.)

iii) We want to solve $H(x)=2x$, i.e

$$\begin{pmatrix}x_1+x_2-x_3\\x_1+2x_2\\x_2+x_3\end{pmatrix} = \begin{pmatrix}2x_1\\2x_2\\2x_3\end{pmatrix}$$

$$\Leftrightarrow \underbrace{\begin{pmatrix}-x_1+x_2-x_3\\x_1\\x_2-x_3\end{pmatrix}}_{\underbrace{\begin{pmatrix}-1&1&-1\\1&0&0\\0&1&-1\end{pmatrix}}_{B}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

(Notice $B=[H]-2I_3$)

We therefore want to solve the system $Bx=0$,

$$(B|0) = \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

$\Rightarrow$ The solutions are $\begin{array}{l} x_1 = 0 \\ x_2 = t \\ x_3 = t \end{array}$ for $t \in \mathbb{R}$.

$\Rightarrow$ All vectors of the form $\begin{pmatrix} 0 \\ t \\ t \end{pmatrix}$ for $t \in \mathbb{R}$ satisfy $H(x) = 2x$.

(Linear Algebra II spoiler: x is called an Eigenvector with Eigenvalue 2 of H)