Homework 6: Basis & Dimension

Deadline: 2nd January, 2022

Exercise 1. (2+4 = 6 Points) Let $U, V \subset \mathbb{R}^n$ be two subspaces. We define their sum by

$$U + V := \{x \in \mathbb{R}^n \mid \text{there exist } u \in U, v \in V \text{ with } x = u + v\}\,.$$

i) Show that $U + V$ is a subspace of $\mathbb{R}^n$.

ii) Show that we have

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)\,.$$

Exercise 2. (6 Points) For $t \in \mathbb{R}$ we define

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ t \end{pmatrix}, \quad v_3 = \begin{pmatrix} t \\ 4 \\ (t-2)^2 \end{pmatrix}$$

and set $V = \operatorname{span}\{v_1, v_2, v_3\}$. For each $t \in \mathbb{R}$ determine a basis of V and calculate its dimension.

Exercise 3. (8 Points) Determine bases for the kernel and the image of the following linear map

$$\begin{aligned} F : \mathbb{R}^5 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto \begin{pmatrix} 0 & 0 & 0 & -2 & 2 \\ -1 & -2 & 1 & 1 & 2 \\ 1 & 2 & -1 & 2 & -5 \end{pmatrix} x\,. \end{aligned}$$

The following exercise is intended to show the basic idea of 3D computer graphics, by showing how to get a 2-dimensional picture (to be shown on a 2-dimensional monitor) from an 3-dimensional object.

Exercise 4. (8 Bonus points)

i) We define the corners of a cube with side length 18 in $\mathbb{R}^3$ by the following set of 8 points:

$$W = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3 \mid w_1, w_2, w_3 \in \{0, 18\} \right\}.$$

Make a drawing of a cube with side length 18 in $\mathbb{R}^3$, i.e. draw the 8 points in the set W and connect two points if they differ just by one entry.

(This just means that you draw a cube like you would usually draw it. "Differ by one entry" just means that these points are on the same edge of the cube.)

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ii) Show that $D = (d_1, d_2, d_3)$ is a basis of $\mathbb{R}^3$, where

$$d_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}.$$

iii) Write each $x \in W$ as a linear combination in the basis D, i.e. for each x find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with

$$x = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 \,.$$

iv) For each $x \in W$ draw the points (λ_1, λ_2) in $\mathbb{R}^2$. Connect two points if the corresponding elements in W just differ by one entry.

Explanation: What you should get in iv) is a drawing of the 3-dimensional cube in 2 dimensions. The basis D somehow describes from which direction you look at the cube. If you replaced the D by the standard basis (e_1, e_2, e_3), you would get a picture of the cube from the top (i.e., just a square). The λ_3, which you did not use for the drawing, describes the distance in the viewing direction.

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くま先生の 簡単数学用語 解説コーナー

Merry (early) Christmas! Another year has passed without notice.... I'm getting old ヽ(；▽；)ノ *sigh*... Anyway, here are today's words, all of them you'll find on this homework.

線形独立(せんけいどくりつ)　　基底(きてい)　　次元(じげん)

These words are: senkei dokuritsu (**linear independence**), kitei (**basis**) and jigen (**dimension**).

(As we learned in HW1) 線形 means "linear".. so naturally 独立(どくりつ) means "independence". Note that this is not only in the context of math; for example, independence day (not the movie!) is 独立記念日(どくりつきねんび).

Similarly, 次元(じげん) here is not only limited to maths; when talking about dimensions in Physics or in everyday life, the same word is used. For example, 3D is 三次元(さんじげん) and 2D is 二次元(にじげん). Speaking of dimensions, in Japan, there is a certain.. thing called 二次元嫁(にじげんよめ)... just saying. Maybe some of you are interested.
... Or maybe you already have one. Anyway, as 線形(せんけい) has been discussed in the HW1, it'll be omitted.

- 独(どく) - This kanji means "**alone**". It refers to how independence means "standing alone (without being under others' influence or control). Other uses of this kanji include 孤独(こどく) (loneliness) and 独身(どくしん) (bachelor).
- 立(りつ) - This kanji means "**to stand**". It refers to how "independence" implies "standing up" for oneself. Common uses of this kanji include the verb 立(た)ちます (to stand), 国立(こくりつ) (State-established) and 私立(しりつ) (Private-established).
- 基(き) - This kanji means "**basic**". It refers to how bases are, in a sense, the "basic blocks" that one can use to "create" any element in the subspace. This kanji is commonly used in everyday life in the word 基本的(きほんてき) (In principle).
- 底(てい) - This kanji means "**bottom**". This kanji further emphasizes the basic nature of bases. While not so common, this kanji is used in 川底(かわそこ) (River bed).
- 次(じ) - This kanji means "**next**". Usually, this kanji is a stand-alone kanji that one would read as 次(つぎ) (also meaning "next").
- 元(げん) - This kanji means "**origin**" or "**former**". While this kanji is not very common in everyday life, it is used when talking about "former title holders", for example "元生徒会長" (Former student council president").

And that's it for today. Merry Christmas! Happy new year!

Well... I hope the new year is better than the last :D