Homework 5: Inverses & Subspaces

Deadline: 12th December, 2021

Exercise 1. (4+4=8 Points)

i) Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 - x_2 \\ -4x_1 + x_2 - x_3 \\ 6x_1 - 2x_2 + x_3 \end{pmatrix},$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} -x_1 + x_3 \\ -4x_1 + x_2 - x_3 \\ 6x_1 - 2x_2 + x_3 \end{pmatrix}.$$

ii) Determine $\ker(F)$ and $\ker(G)$.

Exercise 2. (4+4 = 8 Points)

i) Which of the following subsets are subspaces? Justify your answers.

$$U_1 = \left\{ x \in \mathbb{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0 \right\},$$
$$U_2 = \left\{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \right\},$$
$$U_3 = \{ x \in \mathbb{R}^n \mid Ax = Bx \}, \quad \text{where } A, B \in \mathbb{R}^{m \times n} \text{ are some fixed matrices,}$$
$$U_4 = \{ x \in \mathbb{R}^n \mid x \bullet v = 0 \}, \quad \text{for a fixed } v \in \mathbb{R}^n.$$

ii) Consider the following subspace

$$U = \ker(P_u) = \left\{ x \in \mathbb{R}^3 \mid P_u(x) = 0 \right\}, \quad \text{where } u = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Find vectors $v_1, \ldots, v_m \in \mathbb{R}^3$ with $U = \operatorname{span}\{v_1, \ldots, v_m\}$ and a linear map $H : \mathbb{R}^n \longrightarrow \mathbb{R}^3$ with $\operatorname{im}(H) = U$ for some $n \geq 1$. What are the possible choices of n and what is $\ker(P_u \circ H)$?

Exercise 3. (3+3 = 6 Points)

i) Let $U, V \subset \mathbb{R}^m$ be subspaces. Decide whether the union $U \cup V$ is also a subspace or not.

ii) Let $U, V \subset \mathbb{R}^m$ be subspaces. Decide whether the intersection $U \cap V$ is also a subspace or not.

Version: December 3, 2021

;

くま先生の 簡単数学用語 解説コーナー

The days are getting short, and the nights are getting cold (in Nagoya). Winter has come.

I hope you guys are ready for Professor Bachmann's Linear Algebra Chrismath Challenge!

Today's words are mostly on the subject of the HW: Inverses and Subspaces.

核 (かく)　　像 (ぞう)　　可逆 (かぎゃく)　　部分空間 (ぶぶんくうかん)

These words are: kaku (**kernel**), zou (**image**), kagyaku (**invertible**), and bubun kuukan (**subspace**).

Here's a question: how do we say "invertible matrix" in Japanese?

While I won't give the answer here, interestingly, there are a few Japanese synonyms for "invertible matrix", namely 正則行列 (せいそくぎょうれつ) (lit. regular matrix) and 非特異行列 (ひとくいぎょうれつ) (lit. non-singular matrix).

Note that this time, 像 (ぞう) is omitted due to its inclusion in HW3's part.

- **核 (かく)** - Read as "かく". This kanji usually means "**nuclear**" or "**nucleus**". Uses include 原子核 (げんしかく) (Atomic Nucleus) and 核兵器 (かくへいき) (Nuclear Weapons).
- **可 (か)** - This kanji means "**able**" or "**consent**". It refers to how an invertible matrix is "able" to be inverted. Common uses of this kanji include 可能性 (かのうせい) (Possibility), 許可 (きょか) (permission), and 不可 (ふか) (unable, or "to fail" a course)
- **逆 (ぎゃく)** - This kanji means "**reverse**". It refers to how an inverse matrix is a reverse of the original matrix. One use of this kanji in everyday life is in the expression 逆 (ぎゃく) ですよ (meaning, **no, it's the opposite!**).
- **部 (ぶ)** - This kanji means "**part**". This word is used in 学部 (がくぶ) (Undergraduate). Other uses of this kanji include 部下 (ぶか) (subordinate), 全部 (ぜんぶ) (all), and 部屋 (へや) (room).
- **分 (ぶん)** - This kanji means "**separate**". One word that you might have encountered that uses this kanji is 分 (わ) かる (to understand). Other common uses include 気分 (きぶん) (feeling) and 多分 (たぶん) (probably).
- **空 (くう)** - This kanji means "**space**" or "**sky**". It refers to how subspaces are... spaces! This kanji is very common in everyday life, with its common uses including 空 (そら) (sky), 空港 (くうこう) (airport), 空白 (くうはく) (blank space) and 空手 (からて).
- **間 (かん)** - This kanji means "**interval**". It refers to how subspaces are "intervals" of space. This kanji is also common in everyday life, used in (among others) 時間 (じかん) (time), 間 (あいだ) (between), 瞬間 (しゅんかん) (moment), and 人間 (にんげん) (human).

And that's it for today. Take care, Good Luck, and Have fun (GLHF)!

P.S. In case you didn't guess it yet, "Invertible Matrix" is 可逆行列 (かぎゃくぎょうれつ) (kagyaku gyouretsu).