

1) (14 Points) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ -1 & 2 & 5 \end{pmatrix}$.

- i) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.
- ii) Calculate the matrix BA and decide if BA^n is invertible for any integer $n \geq 1$.
- iii) Determine if $\text{im}(A) \cup \text{im}(B)$ and $\text{im}(A) \cap \text{im}(B)$ are subspaces and, if they are, determine a basis.

2) (12 Points) We define the subspace $U = \text{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}.$$

- i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.
- ii) Calculate the coordinate vectors $[u_1]_B, [u_2]_B, [u_3]_B$ and $[u_4]_B$, where B is the basis from i).
- iii) Calculate an orthonormal basis $F = (f_1, \dots, f_m)$ for U and determine $[u_1]_F, [u_2]_F, [u_3]_F$ and $[u_4]_F$.
- iv) Determine a basis for U^\perp .

3) (12 Points) Set $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and let $C \in \mathbb{R}^{2 \times 2}$ be an arbitrary matrix. Which of the following subsets of \mathbb{R}^2 are subspaces? Justify your answers.

- i) $U_1 = \{x \in \mathbb{R}^2 \mid x \bullet x = x \bullet u\}$.
- ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 - x_2 = 3x_2 \right\}$.
- iii) $U_3 = \{x \in \mathbb{R}^2 \mid Cx = x\}$.
- iv) $U_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \cdot x_2 \geq x_1 \right\}$.

4) (12 Points) Assume we have the following data points

i	1	2	3
x_i	1	2	3
y_i	0	-1	-3

- i) Find the line of best fit for the above data, i.e. find $m, n \in \mathbb{R}$ such that the function $l(x) = mx + n$ minimizes the sum of squares $\sum_{i=1}^3 (l(x_i) - y_i)^2$.
- ii) We define the following linear map

$$H : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and set $V = \text{im}(H)$. Determine a basis of V and for $b = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$ calculate the orthogonal projection $P_V(b)$. Use your result to show that b is not an element in V .

After finishing this exam submit your solution as one pdf file at NUCT at the "Final exam" assignment.