

1) (10 Points) Consider the following linear system

$$\begin{cases} -2x_1 + 4x_2 + x_3 + x_4 = 6 \\ -3x_1 + 6x_2 + x_3 = 7 \\ x_1 - 2x_2 + x_4 = -1 \end{cases} .$$

- i) Find a matrix $A \in \mathbb{R}^{3 \times 4}$ and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.
- ii) Determine the row-reduced echelon forms of the matrices $(A | b)$ and A .
- iii) Find all the solutions to the linear system.
- iv) Calculate the rank of $(A | b)$ and A .
- v) Find all $y \in \mathbb{R}^4$ with $Ay = 2b$ by using your result for iii).

2) (8 Points) Let $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$x \mapsto \begin{pmatrix} u \bullet x \\ 0 \\ x \bullet u \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 2^{x_1+x_2} - 1, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + x_2x_3 \end{pmatrix} .$$

- i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.
- ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map with

$$G \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad G \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} .$$

Determine the matrix of G .

ii) Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a function with

$$F \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad F \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} .$$

Show that F is not a linear map.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \\ x_2 + x_3 \end{pmatrix} .$$

- i) Calculate the image of H .
- ii) Decide if H is injective and/or surjective.
- iii) Find a non-zero vector $v \in \mathbb{R}^3$, such that v is orthogonal to $H(v)$. (Just one explicit vector is enough)

After finishing this exam please send your solution as one pdf file to
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