Linear Algebra I

Fall 2020

§0 Introduction

Linear Algebra = "study of solutions of linear equations."

More precisely: Study of "flat" spaces, called vector spaces / linear spaces.
(e.g. liner, planes, 3dim-space, ...)
and maps between these spaces, called linear maps (e.g. rotations, projections, ...).

In this course we will just consider $\mathbb{R}^n$.
(more general: Linear Algebra II)

- Earliest origins in China more than 2000 years ago.
- Modern form: early 20th century.
- Universally useful in Mathematics, Sciences & Engineering.

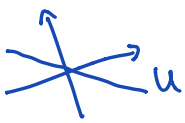
Linear Algebra I roadmap

Linear systems
Matrices, vectors

Week 1–2

$$2x_1 + x_2 = 1$$
$$-x_1 + 3x_2 = -2$$
$$\mathbb{R}^n$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$



$A \in \mathbb{R}^{m \times n}$

$$F(v) = A \cdot v$$

$$F(u+v) = F(u) + F(v)$$
$$F(\lambda \cdot u) = \lambda F(u)$$

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 + x_2 = 0 \right\}$$

$F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Linear maps

Week 3–8

Midterm exam

Subspaces
Kernel & Image

Week 9–10

$$F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$\cup \qquad\qquad \cup$$
$$\operatorname{Ker} F \qquad \operatorname{im} F$$

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Bases
linear independence

Week 11–13

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \Rightarrow \lambda_i = 0 \quad \forall i$$

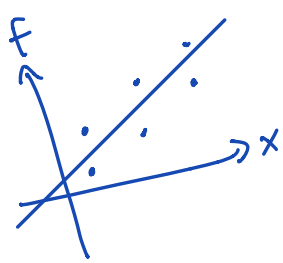
Orthogonality

Week 14–16

Final exam

Linear Algebra II

$$u \bullet v = 0$$



§ 1 Linear systems

$\mathbb{R}$: The set of real numbers, e.g. $1, 2, -1, \frac{1}{3}, \sqrt{2}, \pi, e, \ldots$.
"$b \in \mathbb{R}$": b is an element in $\mathbb{R}$, i.e. b is a real number.

Definition 1.1

i) Linear equation (lin. eq): $a_1 x_1 + \ldots + a_n x_n = b \qquad a_1, \ldots, a_n, b \in \mathbb{R}$

ii) Linear system: A finite collection of lin. eq..

iii) Solution of a linear system: Simultaneous solution to all of its eq..

Goal: Given a linear system find all of its solutions.

Example 1:

$$\begin{cases} x_1 + 3x_2 = 1 \\ -2x_1 + x_2 = 2 \end{cases}$$

② → "Add 2·first eq. to the second"

"This implies" ⟹

$$\begin{cases} x_1 + 3x_2 = 1 \\ 7x_2 = 4 \end{cases}$$

$\left(\frac{1}{7}\right)$ "multiply second eq. by $\frac{1}{7}$."

⟹

$$\begin{cases} x_1 + 3x_2 = 1 \\ x_2 = \frac{4}{7} \end{cases}$$

(−3)

$$\Rightarrow \begin{cases} x_1 = 1 - 3 \cdot \frac{4}{7} = -\frac{5}{7} \\ x_2 = \frac{4}{7} \end{cases}$$

Is $x_1 = -\frac{5}{4}$, $x_2 = \frac{4}{7}$ a solution to the original linear system?

Yes, because the operations work also in reverse.

$$\overset{\textcircled{3}}{\rightarrow}\begin{cases} x_1 = -\frac{5}{7} \\ x_2 = \frac{4}{7} \end{cases} \Rightarrow \overset{\textcircled{7}}{}\begin{cases} x_1 + 3x_2 = 1 \\ x_2 = \frac{4}{7} \end{cases} \Rightarrow \overset{\textcircled{-2}}{\rightarrow}\begin{cases} x_1 + 3x_2 = 1 \\ 7x_2 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + 3x_2 = 1 \\ -2x_1 + x_2 = 2 \end{cases}$$

Therefore:

$$\begin{cases} x_1 + 3x_2 = 1 \\ -2x + x_2 = 2 \end{cases} \Longleftrightarrow \begin{cases} x_1 = -\frac{5}{7} \\ x_2 = \frac{4}{7} \end{cases}$$

iff
"if and only if".

This linear system has exactly one solution.

Example 2

Now consider the following linear system

$$\begin{array}{l} -2 \quad -3 \\ \\ \\ \\ \end{array}\left\{\begin{array}{l} x_1 - 9x_2 - 3x_3 + x_4 = 4 \\ 3x_1 - 2x_2 + x_3 - 2x_4 = 2 \\ 2x_1 + 7x_2 + 4x_3 - 3x_4 = -2 \\ \quad 25x_2 + 9x_3 - 2x_4 = -4 \end{array}\right. \iff \begin{array}{l} -1 \quad -1 \\ \\ \\ \\ \end{array}\left\{\begin{array}{l} x_1 - 9x_2 - 3x_3 + x_4 = 4 \\ 25x_2 + 10x_3 - 5x_4 = -10 \\ 25x_2 + 10x_3 - 5x_4 = -10 \\ 25x_2 + 9x_3 - 2x_4 = -4 \end{array}\right.$$

$$\iff \begin{array}{l} \\ \\ -3 \quad 10 \\ \\ \end{array}\left\{\begin{array}{l} x_1 - 9x_2 - 3x_3 + x_4 = 4 \\ 25x_2 + 10x_3 - 5x_4 = -10 \\ \text{(We can ignore this eq.)} \quad 0 = 0 \\ -x_3 + 3x_4 = 6 \end{array}\right. \iff \begin{array}{l} \\ \frac{1}{25} \\ -1 \end{array}\left\{\begin{array}{l} x_1 - 9x_2 - 8x_4 = -14 \\ 25x_2 + 25x_4 = 50 \\ -x_3 + 3x_4 = 6 \end{array}\right.$$

row-reduced echelon form

$$\iff \begin{array}{l} \\ 9 \\ \\ \end{array}\left\{\begin{array}{l} x_1 - 9x_2 - 8x_4 = -14 \\ x_2 + x_4 = 2 \\ x_3 - 3x_4 = -6 \end{array}\right. \iff \boxed{\left\{\begin{array}{l} x_1 + x_4 = 4 \\ x_2 + x_4 = 2 \\ x_3 - 3x_4 = -6 \end{array}\right.} \quad (*)$$

$$\iff \left\{\begin{array}{l} x_1 = 4 - x_4 \\ x_2 = 2 - x_4 \\ x_3 = -6 + 3x_4 \end{array}\right.$$

The variable x_4 can be choosen arbitrary.

We set $x_4 = t$ for some $t \in \mathbb{R}$.

All the solutions are given by

$$\begin{aligned} x_1 &= 4 - t \quad \text{for } t \in \mathbb{R}. \\ x_2 &= 2 - t \\ x_3 &= -6 + 3t \\ x_4 &= t \end{aligned}$$

Equation (*): Each equation contains a variable that occurs in no other equation: (x_1, x_2, x_3), called pivot variables.

The other variables (x_4) are called free variables.

This equation is in row-reduced echelon form (see HW 1 for a definition).

Later we will define this for matrices.

How to get to this form?

Definition 1.2 Elementary row operations

(R1) Add a multiple of an eq. to another.

(R2) Multiply an eq. with a non-zero number.

(R3) Change the order of the eq.

Since all elementary row operations work in reverse we get:

Proposition 1.3 Applying an elementary row operation to a linear system does not change the set of all solutions of this linear system.

General solutions method (Gaussian elimination)

(Brings a linear system to row-reduced echelon form)

I: Downwards:

1) Make the first eq. contain the first variable (R3)
2) Make the coefficient of this variable = 1 (R2)
3) Eliminate this variable from all other eq. (R1)
4) Iterate with first occuring variable in the remaining eq.

II: Upwards

1) Let x_i be the first variable in the last eq. Eliminate x_i from all other eq. (R1)
2) Go to previous eq. and iterate.

Example 3

(R3)

$$\begin{cases} -x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 - x_3 = 1 \end{cases} \iff \begin{matrix} \\ \\ (-1) \end{matrix} \begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ -x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 = 1 \end{cases}$$

(R1)

$\Longleftrightarrow$

(R2) (R1)

$$\begin{array}{r} \\ -1 \\ \\ \end{array}\begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ -x_3 + x_4 = 0 \\ -2x_3 + x_4 = 1 \end{cases} \Longleftrightarrow \begin{array}{r} \\ 2 \\ \\ \end{array}\begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ x_3 - x_4 = 0 \\ -2x_3 + x_4 = 1 \end{cases}$$

$\Longleftrightarrow$

(R2) From here on upwards (R1)

$$\begin{array}{r} \\ \\ -1 \end{array}\begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ x_3 - x_4 = 0 \\ -x_4 = 1 \end{cases} \Longleftrightarrow \begin{array}{r} \\ 1 \\ 1 \end{array}\begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ x_3 - x_4 = 0 \\ x_4 = -1 \end{cases}$$

(R1)

$$\Longleftrightarrow \begin{array}{r} \\ -1 \\ \\ \end{array}\begin{cases} x_1 + x_2 + x_3 = -1 \\ x_3 = -1 \\ x_4 = -1 \end{cases} \Longleftrightarrow \begin{cases} x_1 + x_2 = 0 \\ x_3 = -1 \\ x_4 = -1 \end{cases}$$

Pivot variables

free variable

The free variables can be choosen arbitrary.
We set $x_2 = t$, $t \in \mathbb{R}$.

All solutions are given by
$$\begin{aligned} x_1 &= -t \\ x_2 &= t \\ x_3 &= -1 \\ x_4 &= -1 \end{aligned} \quad \text{for } t \in \mathbb{R}.$$