Homework 8: Bases & Orthogonality

Deadline: 31th January, 2021

Exercise 1. (6 Points) Let $U = \text{span}\{u_1, u_2, u_3, u_4\} \in \mathbb{R}^4$, where

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 2 \\ 4 \\ -7 \\ 5 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ -5 \\ 7 \\ -1 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of U.

ii) Calculate the coordinate vectors $[u_j]_B \in \mathbb{R}^m$ for $j = 1, 2, 3, 4$.

Let U be a subspace. A basis $(f_1, \dots, f_m)$ of U is called an **orthonormal basis of** U, if for $1 \leq i, j \leq m$

$$f_i \bullet f_j = \begin{cases} 1 & \text{, if } i = j \\ 0 & \text{, if } i \neq j \end{cases}.$$

In other words: The f_j are pairwise orthogonal and they all have length (norm) 1.

Gram-Schmidt algorithm (GSA): Let $B = (b_1, \dots, b_m)$ be an arbitrary basis of a subspace $U \subset \mathbb{R}^n$. The GSA constructs an orthonormal basis $F = (f_1, \dots, f_m)$ of U out of the basis B in the following way:

Step 1: Set $f_1 = \frac{1}{||b_1||} b_1$.

Step l $(2 \leq l \leq m)$: We have constructed orthonormal vectors $f_1, \dots, f_{l-1}$ in the steps before. Now set

$$w_l = b_l - (b_l \bullet f_1) f_1 - \cdots - (b_l \bullet f_{l-1}) f_{l-1} = b_l - \sum_{i=1}^{l-1} (b_l \bullet f_i) f_i$$

and define $f_l = \frac{1}{||w_l||} w_l$.

Exercise 2. (6 Points) We define the following vectors

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

These form a basis $B = (b_1, b_2, b_3)$ of the subspace $U = \text{span}\{b_1, b_2, b_3\} \subset \mathbb{R}^4$ (You do not need to show this). Use the Gram-Schmidt algorithm to construct an orthonormal basis $F = (f_1, f_2, f_3)$ of U from B.

Exercise 3 is on the next page!

Exercise 3. (6 Points) Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on one (non-vertical) line, if there exist $a, b \in \mathbb{R}$ with $ax_j + b = y_j$ for $j = 1, 2, 3$. In other words the linear system

$$\underbrace{\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{=y}$$

has a solution, i.e. $y \in \operatorname{im}(A)$.

i) Show that the points $(0, 1)$, $(1, 3)$ and $(2, 2)$ do <u>not</u> lie on one line.

For ii) - iv) we assume that $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

ii) Calculate an orthonormal basis $F = (f_1, f_2)$ of $\operatorname{im}(A)$ by using the GSA for the columns of A. (Hint: The result becomes nicer if you set b_1 = second column of A and b_2 = first column of A.)

iii) Calculate $z = (y \bullet f_1)f_1 + (y \bullet f_2)f_2$ and show that $z \in \operatorname{im}(A)$.

iv) Solve the linear system $A \begin{pmatrix} a \\ b \end{pmatrix} = z$ and draw the graph of $f(x) = ax + b$ together with the three points in i). Can you interpret the connection between the graph and the points?