

Homework 7: Bases

Deadline: 17th January, 2021

Exercise 1. (6 Points) Let $V \subset \mathbb{R}^n$ be a subspace, $v_1, \dots, v_l \in V$ linearly independent and $V = \text{span}\{w_1, \dots, w_m\}$ for some $w_1, \dots, w_m \in \mathbb{R}^n$. Show that we have $l \leq m$. (Without using Lemma 10.2)

In other words: Show that a space spanned by m vectors can not contain more than m linearly independent vectors.

Exercise 2. (6 Points) For $t \in \mathbb{R}$ we define

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ t \end{pmatrix}, \quad v_3 = \begin{pmatrix} t \\ 4 \\ (t-2)^2 \end{pmatrix}$$

and set $V = \text{span}\{v_1, v_2, v_3\}$. For each $t \in \mathbb{R}$ determine a basis of V and calculate its dimension.

The following exercise is intended to show the basic idea of 3D computer graphics, by showing how to get a 2-dimensional picture (to be shown on a 2-dimensional monitor) from an 3-dimensional object.

Exercise 3. (8 Bonus points)

i) We define the corners of a cube with side length 18 in \mathbb{R}^3 by the following set of 8 points:

$$W = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3 \mid w_1, w_2, w_3 \in \{0, 18\} \right\}.$$

Make a drawing of a cube with side length 18 in \mathbb{R}^3 , i.e. draw the 8 points in the set W and connect two points if they differ just by one entry.

(This just means that you draw a cube like you would usually draw it. "Differ by one entry" just means that these points are on the same edge of the cube.)

ii) Show that $D = (d_1, d_2, d_3)$ is a basis of \mathbb{R}^3 , where

$$d_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}.$$

iii) Write each $x \in W$ as a linear combination in the basis D , i.e. for each x find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with

$$x = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3.$$

iv) For each $x \in W$ draw the points (λ_1, λ_2) in \mathbb{R}^2 . Connect two points if the corresponding elements in W just differ by one entry.

Explanation: What you should get in iv) is a drawing of the 3-dimensional cube in 2 dimensions. The basis D somehow describes from which direction you look at the cube. If you replaced the D by the standard basis (e_1, e_2, e_3) , you would get a picture of the cube from the top (i.e., just a square). The λ_3 , which you did not use for the drawing, describes the distance in the viewing direction.