Homework 3: Functions & Linear maps

Deadline: 8th November, 2020

Exercise 1. (3+4+3=10 Points) We define the following four functions:

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R}^2, \quad x \longmapsto \begin{pmatrix} \cos(x) \\ \sin(x) \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 + x_2 \\ x_1 - x_2 \end{pmatrix},$$

$$f_3 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto 4x - 1,$$

$$f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ x_1 x_2 \end{pmatrix}.$$

i) Calculate the image of each function, i.e. describe $\mathrm{im}(f_j)$ for $j = 1, 2, 3, 4$ as explicit as possible.

ii) Decide for each function if it is injective and/or surjective and/or bijective.

iii) Decide which of the above functions are linear maps.

Justify your answers in ii) and iii).

Exercise 2. (6 Points) Show that there exist a unique linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad G\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

What is the value of $G(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of G.

Exercise 3. (4 Points) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that the following two statements are equivalent:

i) F is injective.

ii) The only solution to $F(x) = 0$ is $x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$.

To show that both statements are equivalent you need to show that i) implies ii) and ii) implies i).

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