1) (12 Points) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

i) Compute the products AB and BA, or explain why they are not defined.

ii) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.

iii) Find all $x \in \mathbb{R}^3$ with $A^T A A^T A A^T A x = 0$. Justify you answer.

2) (12 Points) We define the subspace $U = \text{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$ and $[u_4]_B$, where B is the basis from i).

iii) Find a linear map $G : \mathbb{R}^3 \to \mathbb{R}^3$ with $\text{im}(G) = U$. What is the dimension of $\ker(G)$?

3) (12 Points) Set $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 - 3x_2 = x_1 \right\}$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \text{ is an integer, i.e. } x_1 \in \{\dots, -2, -1, 0, 1, 2, \dots\} \right\}$.

iii) $U_3 = \left\{ x \in \mathbb{R}^2 \mid x \notin \text{span}\{u\} \right\}$.

iv) $U_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet u = x_1 \right\}$.

4) (14 Points) We define the following linear map

$$H : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ -2 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

i) Show that $\dim(\text{im}(H)) = 2$.

ii) Calculate an orthonormal basis (f_1, f_2) for $\text{im}(H)$.

iii) Find a vector $v \in \mathbb{R}^3$, such that $B = (f_1, f_2, v)$ is an orthonormal basis for $\mathbb{R}^3$.

iv) Calculate $[H(x)]_B$ for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

v) Find a $x \in \mathbb{R}^2$ such that $\|H(x) - b\|$ is minimal, where $b = \begin{pmatrix} -5 \\ 1 \\ -1 \end{pmatrix}$.

After finishing this exam please send your solution as **one pdf file** to
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