Tutorial 9 (Linear algebra)

Solutions

Exercise 1

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 1 & -5 & -1 \end{pmatrix}.$$

i)
$$AB = \begin{pmatrix} 3 & 6 & -1 & 7 \\ 1 & 4 & 2 & 5 \\ 3 & 1 & -6 & 2 \end{pmatrix}$$

BA is not defined, since a $m \times n$ matrix can just be multiplied with a $n \times l$ matrix. But $B \in \mathbb{R}^{3\times 4}$ and $A \in \mathbb{R}^{3\times 3}$

ii) B is not invertible because it is not a square matrix.
To check if A is invertible we consider the rref of $(A \mid I_3)$

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 2 & 0 & -2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & -3 & 2 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -4 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 3 & 1 \\ 0 & 1 & 0 & 2 & -2 & -1 \\ 0 & 0 & 1 & 3 & -4 & -1 \end{array}\right) = (I_3 \mid A^{-1}).$$

iii) With $C = A^{-1}B$ we have $AC = \underbrace{AA^{-1}}_{I}B = B$.

$$C = \begin{pmatrix} -2 & 3 & 1 \\ 2 & -2 & -1 \\ 3 & -4 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 1 & -5 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 8 & -9 & -1 \\ -4 & -5 & 9 & 3 \\ -6 & -10 & 11 & 2 \end{pmatrix}.$$

Exercise 2

i) In general: If $u_1, u_2 \in \mathbb{R}^3$ are two linearly independent vectors, then $U = \operatorname{span}\{u_1, u_2\}$ is a subspace with $\dim U = 2$.

Example: $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$U = \operatorname{span}\{u_1, u_2\} = \left\{ \begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, t_1, t_2 \in \mathbb{R} \right\}$.

ii) In general: If (u_1, u_2) is a basis of U, then one can choose $u_3 = u_1 + u_2$.

If $u \in U$, then
$$\begin{aligned} u &= \lambda_1 u_1 + \lambda_2 u_2 \in \operatorname{span}\{u_1, u_2\} \\ &= (\lambda_1 - \lambda_2) u_1 + \lambda_2 u_3 \in \operatorname{span}\{u_1, u_3\} \\ &= (\lambda_2 - \lambda_1) u_2 + \lambda_1 u_3 \in \operatorname{span}\{u_2, u_3\}. \end{aligned}$$

In our example: $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

iii) In general: If $B = (u_1, u_2)$ is a basis of U and $S_B = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix}$, then
$$\begin{aligned} U^\perp &= \{ x \in \mathbb{R}^3 \mid x \cdot u_1 = x \cdot u_2 = 0 \} \\ &= \{ x \in \mathbb{R}^3 \mid \underbrace{S_B^T x = 0}_{} \} \\ &= \ker S_B^T. \end{aligned}$$

→ This will always have a solution of the form $t \cdot w$ for $t \in \mathbb{R}$ and $w \in \mathbb{R}^3$.

In our example: $S_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $S_B^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$S_B^T x = 0 \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right) \rightsquigarrow$ Solutions are $x = t \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{= w}$

$\Rightarrow U^\perp = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. Basis: $\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$.

iv) In general: If (u_1,u_2) is a basis of U and (w) a basis of $U^\perp$, then (u_1,u_2,w) is a basis of $\mathbb{R}^3$.

In our example: $u_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $u_2=\begin{pmatrix}0\\1\\0\end{pmatrix}$, $w=\begin{pmatrix}0\\0\\1\end{pmatrix}$.

v) In general: • If (u_1,u_2) is a basis of U, then we can define $G: \mathbb{R}^3 \to \mathbb{R}^3$

$$x \mapsto \begin{pmatrix} | & | & 0 \\ u_1 & u_2 & 0 \\ | & | & 0\end{pmatrix}.$$

Clearly $\operatorname{im} G = \operatorname{span}\{u_1,u_2\} = U$.

• Let (u_1,u_2) be a basis of U and $w\in\mathbb{R}^3$ such that $\underbrace{(u_1,u_2,w)}_{B}$ is a basis of $\mathbb{R}^3$ (for example the one from iv)).

Then we would like to find a linear map $H:\mathbb{R}^3\to\mathbb{R}^3$ with $[H]_B=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$, because this would mean that for $x=\lambda_1\cdot u_1+\lambda_2\cdot u_2+\lambda_3 w$ we have $H(x)=\lambda_3 w$.

By definition/Proposition in the Lecture (see Tutorial 8) we have

$$[H]_B = S_B^{-1}\,[H]\,S_B \quad \text{where}$$

$S_B=\begin{pmatrix} | & | & | \\ u_1 & u_2 & w \\ | & | & |\end{pmatrix}$. Therefore we need to calculate

$$[H] = \underbrace{S_B}_{\begin{pmatrix} | & | & | \\ u_1 & u_2 & w \\ | & | & |\end{pmatrix}}\ \underbrace{[H]_B}_{\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}}\ S_B^{-1} = \begin{pmatrix}0&0&|\\0&0&w\\0&0&|\end{pmatrix} S_B^{-1}.$$

This will give a linear map $H:\mathbb{R}^3\to\mathbb{R}^3$ with $\operatorname{Ker} H = U$.

In our example: • $G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$x \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x$$

$$\operatorname{im} G = U.$$

• $S_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ i.e. $S_B^{-1} = S_B = I_3$.

$$[H] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

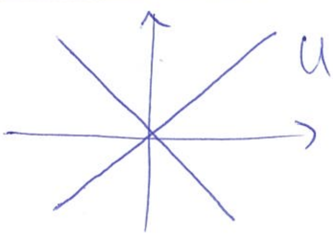
The map $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $x \longmapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ satisfies $\operatorname{Ker} H = U$.

Exercise 3

Recall the definition of subspace:

$U \subset \mathbb{R}^2$ is a subspace if

1) $0 \in U$
2) $u, v \in U \Rightarrow u + v \in U$
3) $\lambda \in \mathbb{R}, u \in U \Rightarrow \lambda \cdot u \in U$.

Conditions 1)	2)	3)	Example
X	X		$U = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1, x_2 \in \mathbb{Z} \}$, $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$. We have $0 \in U$ and $u + v \in U$ if $u, v \in U$, but for example $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in U$ but $\frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin U$.
	X	X	$U = \{\}$ does not contain 0, but 2) and 3) are satisfied.
X		X	$U = \operatorname{span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\} \cup \operatorname{span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$. We have $0 \in U$ and $\lambda u \in U$ for $u \in U$, but $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \notin U$. 

Tutorial 9

Exercise 4

i) To determine a basis of im(F) we consider the rref of [F]

$$[F] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -5 \\ 1 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

pivot positions.

$B = (b_1, b_2)$ with $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is a basis of im(F).

ii) We need to find λ_1, λ_2 with $x = \lambda_1 b_1 + \lambda_2 b_2$:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & t \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & -1 & t-3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & t-2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & t-2 \end{array}\right).$$

We get a solution for $t = 2$ given by $\lambda_1 = 2$ and $\lambda_2 = 1$.

$t=2$:

$$\Rightarrow \quad [x]_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}. \qquad \left(\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 2 \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{b_1} + 1 \cdot \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{b_2} \right).$$

iii) We use the Gram-Schmidt algorithm to get an ONB $E = (f_1, f_2)$:

$$f_1 = \frac{1}{\|b_1\|} b_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$w_2 = b_2 - (f_1 \cdot b_2) f_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{3} \cdot 0 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(b_2 and b_1 are already orthogonal).

$$f_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

$$E = \left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)$$ is an ONB for im F.

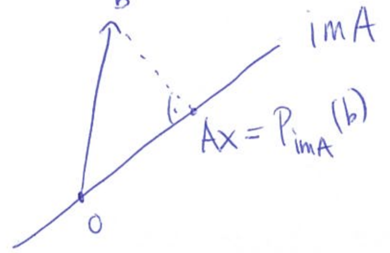
Exercise 5

i) For example $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

In this case $Ax = b$ has no solutions, since

$$(A|b) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 0 & 0 & 2 \end{array}\right).$$

ii) In general: We saw that we can minimize $\|Ax-b\|$ by choosing $Ax = P_{\text{im}A}(b)$, i.e. the orthogonal projection of b onto $\text{im}\,A$ should be Ax. This can be done by solving



$$A^T A x = A^T b. \quad \text{(see last Lecture).}$$

In our example:

$$A^T A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 8 & 11 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \end{pmatrix}.$$

Want to solve $\begin{pmatrix} 6 & 8 \\ 8 & 11 \end{pmatrix} x = \begin{pmatrix} 8 \\ 10 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 6 & 8 & 8 \\ 8 & 11 & 10 \end{array}\right) \sim \left(\begin{array}{cc|c} 6 & 8 & 8 \\ 2 & 3 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & -1 & 2 \\ 2 & 3 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & \frac{3}{2} & 1 \\ 0 & -1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \end{array}\right)$$

$\Rightarrow x = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ minimizes $\|Ax - b\|$.

$$\left(Ax = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \rightsquigarrow \|Ax - b\| = \left\| \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\| = \sqrt{2} \right)$$