Tutorial 8: Change of basis & coordinates

Let $B = (b_1, \dots, b_n)$ be a basis of $\mathbb{R}^n$. The **coordinate map** is defined by

$$c_B : \mathbb{R}^n \longrightarrow \mathbb{R}^n \,,$$
$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \longmapsto \lambda_1 b_1 + \dots \lambda_n b_n \,.$$

The **change-of-basis matrix** S_B associated with B is given by the matrix of c_B:

$$S_B := [c_B] = \begin{pmatrix} | & & | \\ b_1 & \dots & b_n \\ | & & | \end{pmatrix} .$$

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. The **matrix of F with respect to** B is the matrix

$$[F]_B := [c_B^{-1} \circ F \circ c_B] = S_B^{-1} \cdot [F] \cdot S_B \,.$$

Recall, that the coordinate vector of an element $x \in \mathbb{R}^n$ is given by $[x]_B := c_B^{-1}(x) = S_B^{-1} x$.

Exercise 1. We define the vectors

$$b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

and the linear map

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

i) Calculate $F(v_1) = [F]v_1$ and $F(v_2) = [F]v_2$.

ii) Show that $B = (b_1, b_2)$ is a basis of $\mathbb{R}^2$.

iii) Calculate S_B^{-1} and $[F]_B$.

iv) Calculate $w_1 = [v_1]_B$ and $w_2 = [v_2]_B$. (the coordinates of the vectors v_1 and v_2 in B)

v) Calculate $[F]_B w_1$ and $[F]_B w_2$.

vi) Draw two diagrams: In the first draw $w_1, w_2, [F]_B w_1, [F]_B w_2$ and in the second draw $v_1, v_2, [F]v_1, [F]v_2$.

vii) What is the relationship between $[F]_B w_1$ and $[F]v_1$ and what is the relationship between $[F]_B w_2$ and $[F]v_2$?