## Exercise 1

1) $\{x_1 = 1$ Solution: $x_1 = 1$

2) $\{x_1 + x_2 = 0$ Solution: $x_1 = -t$, $x_2 = t$ for $t \in \mathbb{R}$

3) $\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 0 \end{cases}$ no solution

## Exercise 2

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ is <u>not</u> on rref

$(1\ 2\ 3)$ is on rref

$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is on rref

$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ — " —

$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ — " —

## Exercise 3

i) $A = \begin{pmatrix} 1 & 4 & 7 & 2 \\ 2 & 5 & 8 & 1 \\ 3 & 6 & 10 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

ii) $(A|b) = \left(\begin{array}{cccc|c} 1 & 4 & 7 & 2 & 1 \\ 2 & 5 & 8 & 1 & 2 \\ 3 & 6 & 10 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 4 & 7 & 2 & 1 \\ 0 & -3 & -6 & -3 & 0 \\ 0 & -6 & -11 & -5 & -2 \end{array}\right)$

$\sim \left(\begin{array}{cccc|c} 1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -6 & -11 & -5 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 4 & 7 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 4 & 0 & -5 & 15 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right)$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & -2 \end{array}\right) = \text{rref}(A|b).$$

iii) Using ii) we get infinitely many solutions. We have one free variable and set $x_4 = t$ for $t \in \mathbb{R}$.

Solutions:
$$\begin{aligned} x_1 &= -1 + t \\ x_2 &= 4 + t \\ x_3 &= -2 - t \\ x_4 &= t \end{aligned}$$

iv) By ii) we have $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and therefore

$rk(A) = 3$ and $rk(A|b) = 3$.

## Exercise 4

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ : $F_A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 \\ 0 \end{pmatrix}$, $F_A : \mathbb{R}^2 \to \mathbb{R}^3$

We have $\text{im}(F_A) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, y_3 = 0 \right\}$, because for each $y_1, y_2 \in \mathbb{R}$ we have $F_A\begin{pmatrix} y_1 - y_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}$.

$\Rightarrow$ $F_A$ is <u>not</u> surjective (because $\text{im}(F_A) \neq \mathbb{R}^3$.)

$\Rightarrow$ $F_A$ is injective, because $x = \begin{pmatrix} y_1 - y_2 \\ y_2 \end{pmatrix}$ is the only vector with $F(x) = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}$.

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$A = (1\ 2\ 3)$: $F_A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 + 2x_2 + 3x_3$, $F_A : \mathbb{R}^3 \to \mathbb{R}$

$\text{im}(F_A) = \mathbb{R} \Rightarrow F_A$ is surjective.

$F_A\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = F_A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$ but $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow F_A$ is <u>not</u> injective.

$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ : $F_A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 - x_3 \\ x_2 + 3x_3 \\ x_4 \end{pmatrix}$, $F_A : \mathbb{R}^4 \to \mathbb{R}^3$

$Ax = F_A(x) = b$ for any $b \in \mathbb{R}^4$ has infinitely many solutions.

$\Rightarrow \operatorname{Im}(F_A) = \mathbb{R}^3 \Rightarrow F_A$ is surjective.

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = F_A\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = F_A\begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix}$ but $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} \Rightarrow F_A$ is <u>not</u> injective

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$A = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ : $F_A(x_1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $F_A : \mathbb{R}^1 \to \mathbb{R}^3$

$\operatorname{im}(F_A) = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\} \neq \mathbb{R}^3 \Rightarrow F_A$ is <u>not</u> ~~injec~~ surjective.

$F_A(1) = F_A(0) \Rightarrow F_A$ is <u>not</u> injective.

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$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ : $F_A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_4 \\ x_3 + x_4 \end{pmatrix}$, $F_A : \mathbb{R}^4 \to \mathbb{R}^2$

$F_A(x) = Ax = b$ has infinitely solutions for any $b \in \mathbb{R}^2$.

$\Rightarrow F_A$ is surjective but <u>not</u> injective. $\operatorname{im}(F_A) = \mathbb{R}^2$

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$A = \begin{pmatrix} 1 & 4 & 7 & 2 \\ 2 & 5 & 8 & 1 \\ 3 & 6 & 10 & 1 \end{pmatrix}$ : $F_A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 4x_2 + 7x_3 + 2x_4 \\ 2x_1 + 5x_2 + 8x_3 + x_4 \\ 3x_1 + 6x_2 + 10x_3 + x_4 \end{pmatrix}$, $F_A : \mathbb{R}^4 \to \mathbb{R}^3$

By Ex. 3 we know that also $Ax = F_A(x) = b$ has infinitidy many solutions for any $b \in \mathbb{R}^3$

$\Rightarrow F_A$ is also surjective but <u>not</u> injective.

## Exercise 5

i) We need to find all vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ with $x \cdot v = 0$. (definition of "orthogonal").

$$x \cdot v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = x_1 + 2x_2 + 3x_3 .$$

The condition $x \cdot v = 0$ is therefore equivalent to

$$x_1 + 2x_2 + 3x_3 = 0 .$$

This is a linear equation with infinitely many solutions and we set $x_2 = t_1$ and $x_3 = t_2$ for $t_1, t_2 \in \mathbb{R}$ and get

$$x_1 = -2t_1 - 3t_2$$
$$x_2 = t_1$$
$$x_3 = t_2 .$$

So all vectors x orthogonal to v are given by

$$x = t_1 \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} . \qquad (*)$$

ii) We define the linear map $G: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 - 3x_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

We see that $\mathrm{im}(G)$ gives exactly all vectors of the form $(*)$:

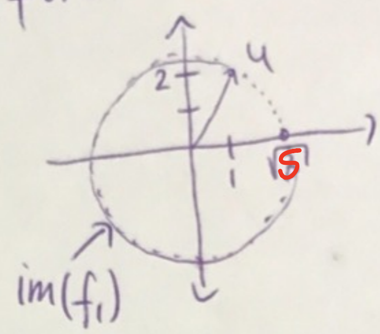
$$\mathrm{im}(G) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, y_1 = -2y_2 - 3y_3 \right\}$$

## Exercise 6

i) The map $f_1$ rotates the fixed vector $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$f_1(x)$ is $u$ rotated by an angle $x$.

We have $\|u\| = \sqrt{5}$ and therefore the image of $f_1$ is a circle with radius $\sqrt{5}$.




$\Rightarrow f_1$ is <u>not</u> surjective.

$f_1$ is also <u>not</u> injective because $f_1(0) = f_1(2\pi) = u$.

ii) $f_1$ is <u>not</u> linear: $f_1(0 \cdot 0) = f_1(0) = rot_0(u) = u$

but $0 \cdot f_1(0) = 0 \cdot u = 0$

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$f_2$ is linear: $f_2(x) = f_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} u \cdot x \\ (u \cdot u)(x \cdot u) \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 5(x_1 + 2x_2) \end{pmatrix}$

$\left( \begin{array}{l} u \cdot \overbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{x} = x_1 + 2x_2 \\ u \cdot u = 1 + 4 = 5 \end{array} \right)$

$$= \begin{pmatrix} x_1 + 2x_2 \\ 5x_1 + 10x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 5 & 10 \end{pmatrix}}_{[f_2]} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

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$f_3$ is <u>not</u> linear: $f_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right) = f_3\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\neq$$

$$f_3\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + f_3\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

$f_4$ is linear: $f_4\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2x_2-x_1\\2x_1\\4x_1+x_2\end{pmatrix} = \underbrace{\begin{pmatrix}-1&2\\2&0\\4&1\end{pmatrix}}_{[f_4]}\begin{pmatrix}x_1\\x_2\end{pmatrix}$

(because we can write $f_4(x)$ as matrix $\cdot\, x$)

Exercise 7 To determine the matrix of $G$ we need to calculate $G\begin{pmatrix}1\\0\end{pmatrix}$ and $G\begin{pmatrix}0\\1\end{pmatrix}$.

$$\underset{G(e_1)}{\overset{\shortparallel}{G\begin{pmatrix}1\\0\end{pmatrix}}} \qquad \underset{G(e_2)}{\overset{\shortparallel}{G\begin{pmatrix}0\\1\end{pmatrix}}}$$

Since $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}$

and $\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}$, we have

$$G\begin{pmatrix}1\\0\end{pmatrix} = G\left(\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\2\end{pmatrix}\right) \underset{\substack{\uparrow\\ G \text{ is linear}}}{=} G\begin{pmatrix}1\\-1\end{pmatrix} + \frac{1}{2}G\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}3\\3\\4\end{pmatrix}$$

and

$$G\begin{pmatrix}0\\1\end{pmatrix} = G\left(\frac{1}{2}\cdot\begin{pmatrix}0\\2\end{pmatrix}\right) = \frac{1}{2}G\begin{pmatrix}0\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}4\\2\\2\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}.$$

The matrix of $G$ is therefore:

$$[G] = \begin{pmatrix}| & | \\ G(e_1) & G(e_2)\\ | & |\end{pmatrix} = \begin{pmatrix}3&2\\3&1\\4&1\end{pmatrix}$$

$$G\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3&2\\3&1\\4&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}5\\4\\5\end{pmatrix}.$$