Tutorial 4: Linear maps

A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear map**, if for all $u, v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ we have

i) $F(u+v) = F(u) + F(v)$,

ii) $F(\lambda u) = \lambda F(u)$.

Exercise 1. Which of the following functions are linear maps?

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sin(x)\,,$$

$$f_2 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^2 + 1\,,$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{pmatrix}\,,$$

$$f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ x_1^2 + x_2 \end{pmatrix}\,.$$

<u>**Theorem:**</u> For any linear map $F : \mathbb{R}^n \to \mathbb{R}^m$ there exist a unique matrix $[F] \in \mathbb{R}^{m \times n}$, such that we have for all $x \in \mathbb{R}^n$

$$F(x) = [F]x\,.$$

(Here the left-hand side is the evaluation of the function F at x and the right-hand side is the multiplication of the matrix $[F]$ with x.)

$[F]$ is called **the matrix of** F.

Exercise 2. Show that there exist a <u>unique</u> linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

What is the value of $T(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of T.

Version: October 28, 2019