

Tutorial 3: Matrices & Functions

Exercise 1. Calculate the rank of the following matrices

$$B = \begin{pmatrix} 3 & 3 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & -1 \end{pmatrix}, \quad D = (1 \ 2 \ 3),$$
$$E = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Exercise 2. Let $A \in \mathbb{R}^{m \times n}$ be a matrix and $b \in \mathbb{R}^n$ be a vector.

- i) What are the possible values of $\text{rk}(A)$ (the rank of A) for a given m and n ?
- ii) How many solutions can we have for the equation $Ax = b$ in each of the following cases?
 - 1) $m = n$,
 - 2) $m < n$,
 - 3) $m > n$.

Let X and Y be two sets.

- A **function** $f : X \rightarrow Y$ is a rule, assigning to each element $x \in X$ an element $f(x) \in Y$. This is also sometimes denoted by

$$f : X \longrightarrow Y$$
$$x \longmapsto f(x).$$

- For a function $f : X \rightarrow Y$ the **image of f** is defined by

$$\text{im}(f) = f(X) = \{y \in Y \mid \exists x \in X : y = f(x)\} \subset Y.$$

Exercise 3. Calculate the image of the following functions:

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto \sin(x),$$

$$f_2 : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2 + 1,$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{pmatrix},$$

$$f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}.$$

A function $f : X \rightarrow Y$ is called

- i) **injective** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. ($x_1, x_2 \in X$)
- ii) **surjective** if $\text{im}(f) = Y$.
- iii) **bijective** if it is injective and surjective.

Exercise 4. For $X = \mathbb{R}$ try to find an example of a function $f : X \rightarrow X$, which is

- i) not injective and not surjective.
- ii) injective but not surjective.
- iii) not injective but surjective.
- iv) bijective.

Do the same for $X = \{1, 2, 3\}$. Is it possible to find examples for all cases?

Exercise 5. We define for a matrix $A \in \mathbb{R}^{m \times n}$ the function

$$\begin{aligned} f : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto Ax. \end{aligned}$$

Find again examples of m , n and A for the cases i),ii),iii) and iv) as in Exercise 4.