Linear Algebra I, Fall 2019

Midterm Exam

We consider the following linear system:

1)
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + 5x_4 &= 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= 7 \end{aligned} \qquad (*)$$

i) With $A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 5 \\ 7 \end{pmatrix}$ the solutions $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ of $Ax = b$ are the same as the solutions of $(*)$.

ii)
$$(A|b) = \left(\begin{array}{cccc|c} 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \end{array}\right) \sim \overset{\textcircled{-3}\,\textcircled{-2}}{} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \end{array}\right)$$

$$\sim \overset{\textcircled{-2}\,\textcircled{-1}}{} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -2 & -4 & -6 & -8 \end{array}\right) \sim \overset{\textcircled{-2}}{} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \underbrace{\left(\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right.}_{\mathrm{rref}(A).}\left.\begin{array}{|c} -3 \\ 4 \\ 0 \end{array}\right) = \mathrm{rref}(A|b).$$

iii) The solutions are given by

$$\begin{aligned} x_1 &= -3 + t_1 + 2t_2 \\ x_2 &= 4 - 2t_1 - 3t_2 \\ x_3 &= t_1 \\ x_4 &= t_2 \end{aligned}$$

with $t_1, t_2 \in \mathbb{R}$.

iv) The ranks are given by $\mathrm{rk}(A|b) = \mathrm{rk}(A) = 2$.

v) There are no solutions of $Ax = c$ for $c = \begin{pmatrix} 6 \\ 5 \\ 6 \end{pmatrix}$, since

$$(A|c) = \left(\begin{array}{cccc|c} 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 6 \end{array}\right) \sim \begin{matrix}\textcircled{\scriptsize -3}\ \textcircled{\scriptsize -2}\end{matrix} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 6 \end{array}\right)$$

$$\sim \begin{matrix}\textcircled{\scriptsize -2}\ \textcircled{\scriptsize -1}\end{matrix} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -2 & -4 & -6 & -9 \end{array}\right) \sim \begin{matrix}\textcircled{\scriptsize -2} \\ \textcircled{\scriptsize -1}\end{matrix} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & \underset{1}{\cancel{}} \end{array}\right) = \text{rref}(A|c) .$$

Here $\underset{\overset{\shortparallel}{3}}{rk(A|b)} > rk(A) = 2$ and therefore there are no solutions.

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2) i) $\underline{f_1}$:

We have

$$f_1(x) = f_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (u \circ x)u + x = \Bigg(\overbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \circ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}^{2x_1 + x_2}\Bigg)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= (2x_1 + x_2)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_1 + 2x_2 \\ 2x_1 + x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5x_1 + 2x_2 \\ 2x_1 + 2x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

$\Rightarrow$ f_1 is linear and $[f_1] = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$.

$\underline{f_2}$: We have $f_2(0) = \begin{pmatrix} 2\cos(0) \\ \sin(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Therefore $f_2(0 + 0) \neq f_2(0) + f_2(0)$.

This means f_2 is <u>not</u> linear.

$\underline{f_3}$: $f_3(x) = f_3\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \frac{x \bullet x}{u \bullet u} u = \frac{x_1^2+x_2^2+x_3^2}{1^2+2^2}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2x_1^2+2x_2^2+2x_3^2\\x_1^2+x_2^2+x_3^2\end{pmatrix}$

We have $f\Big(\underset{\underset{2\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\|}}{\begin{pmatrix}2\\0\\0\end{pmatrix}}\Big) = \frac{1}{5}\begin{pmatrix}8\\4\end{pmatrix}$,

but $2 \cdot f\Big(\begin{pmatrix}1\\0\\0\end{pmatrix}\Big) = 2 \cdot \frac{1}{5}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{5}\begin{pmatrix}4\\2\end{pmatrix}$.

Therefore $2 f\begin{pmatrix}1\\0\\0\end{pmatrix} \neq f\Big(2 \cdot \begin{pmatrix}1\\0\\0\end{pmatrix}\Big)$.

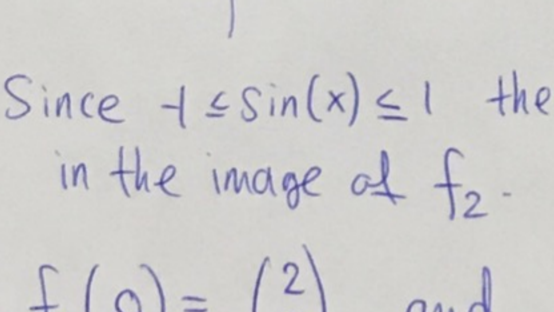
$\Rightarrow$ f_3 is <u>not</u> linear.

$\underline{f_4}$: We have

$$f(x) = f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\x_1+2x_2+3x_3\\x_1-x_3\\x_2\end{pmatrix} = \begin{pmatrix}0&0&0\\1&2&3\\1&0&-1\\0&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}.$$

$\Rightarrow$ f_4 is linear with $[f_4] = \begin{pmatrix}0&0&0\\1&2&3\\1&0&-1\\0&1&0\end{pmatrix}$.

ii)



The image of f_2 is an ellipse.

- Since $-1 \leq \sin(x) \leq 1$ the vector $\begin{pmatrix}0\\2\end{pmatrix}$ can not be in the image of f_2. $\Rightarrow$ f_2 is <u>not</u> surjective.
- $f_2(0) = \begin{pmatrix}2\\0\end{pmatrix}$ and $f_2(2\pi) = \begin{pmatrix}2\\0\end{pmatrix}$, but $0 \neq 2\pi$. $\Rightarrow$ f_2 is <u>not</u> injective.

3) We first try to find for $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2$ numbers $a,b\in\mathbb{R}$ with

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=x=a\cdot\begin{pmatrix}1\\1\end{pmatrix}+b\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\overset{(-1)}{\rightarrow}\left(\begin{array}{cc|c}1&1&x_1\\1&2&x_2\end{array}\right)\sim\overset{\uparrow}{(-1)}\left(\begin{array}{cc|c}1&1&x_1\\0&1&x_2-x_1\end{array}\right)\sim\left(\begin{array}{cc|c}1&0&2x_1-x_2\\0&1&x_2-x_1\end{array}\right).$$

Therefore $\begin{pmatrix}x_1\\x_2\end{pmatrix}=(2x_1-x_2)\begin{pmatrix}1\\1\end{pmatrix}+(x_2-x_1)\begin{pmatrix}1\\2\end{pmatrix}.$

In particular $\begin{pmatrix}1\\0\end{pmatrix}=2\cdot\begin{pmatrix}1\\1\end{pmatrix}+(-1)\begin{pmatrix}1\\2\end{pmatrix}$

$$\begin{pmatrix}0\\1\end{pmatrix}=(-1)\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}.$$

This means $G\begin{pmatrix}1\\0\end{pmatrix}=G\left(2\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}1\\2\end{pmatrix}\right)$

$$=2G\begin{pmatrix}1\\1\end{pmatrix}-G\begin{pmatrix}1\\2\end{pmatrix}=2\begin{pmatrix}-1\\1\end{pmatrix}-\begin{pmatrix}3\\4\end{pmatrix}$$

$$=\begin{pmatrix}-2\\2\end{pmatrix}-\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}-5\\-2\end{pmatrix}$$

$$G\begin{pmatrix}0\\1\end{pmatrix}=G\left(-\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}\right)=-G\begin{pmatrix}1\\1\end{pmatrix}+G\begin{pmatrix}1\\2\end{pmatrix}$$

$$=\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}.$$

i)

The matrix of G is given by

$$[G]=\begin{pmatrix}|&|\\G\begin{pmatrix}1\\0\end{pmatrix}&G\begin{pmatrix}0\\1\end{pmatrix}\\|&|\end{pmatrix}=\begin{pmatrix}-5&4\\-2&3\end{pmatrix}.$$

ii) The matrix of $G\circ G$ is given by

$$[G\circ G]=[G]\cdot[G]=\begin{pmatrix}-5&4\\-2&3\end{pmatrix}\begin{pmatrix}-5&4\\-2&3\end{pmatrix}=\begin{pmatrix}17&-8\\4&1\end{pmatrix}.$$

4) i) $$H\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{[H].}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

To calculate the image of H, we need to find all $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3$, such that $[H]x = y$ has a solution.

$$([H]|y) = \begin{array}{l} \scriptstyle (-1) \\ \\ \scriptstyle \hookrightarrow \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 1 & 1 & 1 & y_3 \end{array}\right) \sim \begin{array}{l} \\ \scriptstyle (-1) \\ \scriptstyle \hookrightarrow \end{array}\left(\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 1 & 1 & y_3 - y_1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 0 & 0 & y_3 - y_1 - y_2 \end{array}\right).$$

$\Rightarrow$ We just have a solution in the case $y_3 = y_1 + y_2$.

$$\Rightarrow \quad \operatorname{im}(H) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y_3 = y_1 + y_2 \right\}.$$

ii) By i) we know that $\operatorname{im}(H) \neq \mathbb{R}^3$ and therefore H is <u>not</u> surjective.

For $y = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ the equation $[H]x = y$ has solutions

$$x_1 = 1$$
$$x_2 = 1 - t \qquad \text{for } t \in \mathbb{R}.$$
$$x_3 = t$$

Choosing $t = 0, 1$ we get

$$H\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = H\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

and therefore H is not injective.

iii) A vector $y \in \operatorname{im}(H)$ has the form $y = \begin{pmatrix} y_1 \\ y_2 \\ y_1 + y_2 \end{pmatrix}$ for $y_1, y_2 \in \mathbb{R}$.
The vector $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is orthogonal to y if $v \cdot y = 0$.

$$v \cdot y = v_1 y_1 + v_2 y_2 + v_3 (y_1 + y_2) = (v_1 + v_3) y_1 + (v_2 + v_3) y_2.$$

We get the linear system $v_1 + v_3 = 0$, $v_2 + v_3 = 0$, which has the solution
$$v = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} \text{ for } t \in \mathbb{R}.$$

These are all vectors which are orthogonal to all vectors in $\operatorname{im}(H)$