

1) (10 Points) Consider the following linear system

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 + 5x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 5 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 = 7 \end{cases}.$$

- i) Find a matrix $A \in \mathbb{R}^{3 \times 4}$ and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.
- ii) Determine the row-reduced echelon forms of the matrices $(A | b)$ and A .
- iii) Find all the solutions to the linear system.
- iv) Calculate the rank of $(A | b)$ and A .
- v) Find a vector $c \in \mathbb{R}^3$, such that $Ax = c$ has no solutions. Calculate the rank of $(A | c)$.

2) (10 Points) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad f_2 : \mathbb{R} \longrightarrow \mathbb{R}^2$$

$$x \longmapsto (u \bullet x)u + x, \quad x \longmapsto \begin{pmatrix} 2 \cos(x) \\ \sin(x) \end{pmatrix},$$

$$f_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \quad f_4 : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$

$$x \longmapsto \frac{x \bullet x}{u \bullet u} u, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ x_1 + 2x_2 + 3x_3 \\ x_1 - x_3 \\ x_2 \end{pmatrix}.$$

- i) Which of the above functions f_1, f_2, f_3, f_4 are linear maps? For each one that is linear, determine its matrix.
- ii) Draw a picture of the image of f_2 . Is f_2 injective and/or surjective?

3) (6 Points) Let $G : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map with

$$G \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad G \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

- i) Determine the matrix of G .
- ii) Determine the matrix of $G \circ G$.

4) (6 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

- i) Calculate the image of H .
- ii) Decide if H is injective and/or surjective.
- iii) Find all vectors $v \in \mathbb{R}^3$, which are orthogonal to all vectors in the image of H .

Do not turn this page before told to do so!

- This exam consists of four problems, with a total score of 32 points.
- Time: 90 minutes (10:30 - 12:00).
- All solutions should be clear answers to the questions asked. Justify your answers!
- According to Nagoya University student discipline rules, cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the term.
- Materials allowed: Writing material only.
- Do not use your own paper. Writing paper will be provided.
- You can not leave the room during the exam, but you can hand in your exam before the time is over. If you do so, please leave the room quietly!
- Do not forget to write your name on each piece of paper you hand in!

Good luck!