Homework 6 (Solution)

Exercise 1 $b_1 = \begin{pmatrix}1\\2\\0\\3\end{pmatrix}$, $b_2 = \begin{pmatrix}4\\0\\5\\8\end{pmatrix}$, $b_3 = \begin{pmatrix}8\\-1\\5\\6\end{pmatrix}$.

Step 1: $f_1 = \frac{1}{\|b_1\|} b_1 = \frac{1}{\sqrt{14}} \begin{pmatrix}1\\2\\0\\3\end{pmatrix}$.

Step 2: $w_2 = b_2 - (f_1 \cdot b_2) f_1 = \begin{pmatrix}4\\0\\5\\8\end{pmatrix} - \frac{1}{14} \cdot \overbrace{(4+3\cdot 8)}^{28} \begin{pmatrix}1\\2\\0\\3\end{pmatrix} = \begin{pmatrix}4\\0\\5\\8\end{pmatrix} - \begin{pmatrix}2\\4\\0\\6\end{pmatrix}$

$= \begin{pmatrix}2\\-4\\5\\2\end{pmatrix}$

$f_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{7} \begin{pmatrix}2\\-4\\5\\2\end{pmatrix}$.

Step 3: $w_3 = b_3 - (f_1 \cdot b_3) f_1 - (f_2 \cdot b_3) f_2$

$= \begin{pmatrix}8\\-1\\5\\6\end{pmatrix} - \frac{28}{14} \begin{pmatrix}1\\2\\0\\3\end{pmatrix} - \frac{49}{49} \begin{pmatrix}2\\-4\\5\\2\end{pmatrix} = \begin{pmatrix}8\\-1\\5\\6\end{pmatrix} - \begin{pmatrix}2\\4\\0\\6\end{pmatrix} - \begin{pmatrix}2\\-4\\5\\2\end{pmatrix} = \begin{pmatrix}4\\1\\0\\-2\end{pmatrix}$.

$f_3 = \frac{1}{\|w_3\|} w_3 = \frac{1}{\sqrt{21}} \begin{pmatrix}4\\1\\0\\-2\end{pmatrix}$.

An orthonormal basis of U is given by

$$F = \left(\frac{1}{\sqrt{14}} \begin{pmatrix}1\\2\\0\\3\end{pmatrix}, \frac{1}{7} \begin{pmatrix}2\\-4\\5\\2\end{pmatrix}, \frac{1}{\sqrt{21}} \begin{pmatrix}4\\1\\0\\-2\end{pmatrix} \right).$$

Exercise 2 i)

(Fast solution: Let $(b_1,\dots,b_m)$ be a basis of U and $A=\begin{pmatrix} | & & | \\ b_1 & \cdots & b_m \\ | & & | \end{pmatrix}$.
Then $U^\perp = \ker A^T$ and therefore $U^\perp$ is a subspace.)

To check if $U^\perp$ is a subspace we need to check 3 conditions:

1) Since $0 \cdot u = 0$ for any $u \in U$, we have $0 \in U^\perp$.

2) Let $x, y \in U^\perp$, i.e $x \cdot u = y \cdot u = 0 \quad \forall u \in U$.
Then $(x+y)\cdot u = x\cdot u + y\cdot u = 0+0=0$ and
therefore $x+y \in U^\perp$.

3) Let $x \in U^\perp$ and $\lambda \in \mathbb{R}$. Then $(\lambda x)\cdot u = \lambda (x\cdot u) = \lambda \cdot 0 = 0$
and therefore $\lambda x \in U^\perp$.

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ii) Let $A = \begin{pmatrix} | & | & | \\ b_1 & b_2 & b_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 4 & 8 \\ 2 & 0 & 1 \\ 0 & 5 & 5 \\ 3 & 8 & 6 \end{pmatrix}$. The elements in $U^\perp$ are orthogonal to b_1, b_2, b_3, i.e $x \in U^\perp \Leftrightarrow x\cdot b_1 = x\cdot b_2 = x\cdot b_3 = 0$.

This means that $U^\perp = \ker A^T$, where $A^T = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 4 & 0 & 5 & 8 \\ 8 & 1 & 5 & 6 \end{pmatrix}$.

$$A^T = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 4 & 0 & 5 & 8 \\ 8 & 1 & 5 & 6 \end{pmatrix} \overset{(-8),(-4)}{\sim} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 5 & -4 \\ 0 & -15 & 5 & -18 \end{pmatrix} \overset{(-2)}{\sim} \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 5 & -4 \\ 0 & 1 & -5 & -10 \end{pmatrix} \overset{(8)}{\sim}$$

$$\sim \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & -35 & -84 \\ 0 & 1 & -5 & -10 \end{pmatrix} \overset{(2)}{\sim} -\frac{1}{35}\begin{pmatrix} 1 & 0 & 10 & 23 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & -35 & -84 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 10 & 23 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & \frac{12}{5} \end{pmatrix} \overset{(-10),(5)}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{12}{5} \end{pmatrix} \Rightarrow x \in \ker A^T \Leftrightarrow x = t \cdot \begin{pmatrix} 1 \\ -2 \\ -\frac{12}{5} \\ 1 \end{pmatrix} \text{ for } t \in \mathbb{R}.$$

A basis for $U^\perp$ is therefore (b_1) with.

$(t=5) \quad b_1 = \begin{pmatrix} 5 \\ -10 \\ -12 \\ 5 \end{pmatrix}$

Exercise 3

i) We need to show that $Ax = b$ with $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ has no solution.

$$(A|y) \overset{(2)}{\sim} \left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \end{array}\right) \sim \overset{(1)}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & -4 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{array}\right)$$

No solutions.

ii) Let $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, then $B = (b_1, b_2)$ is a basis of $\operatorname{im} A$. Using the Gram-Schmidt algorithm we obtain

$$f_1 = \frac{1}{\|b_1\|} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$w_2 = b_2 - (b_2 \circ f_1) f_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{1}{3}(1+3) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

$$f_2 = \frac{1}{\|w_2\|} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

iii) $F = (f_1, f_2)$ is an ONB of $\operatorname{im} A$.

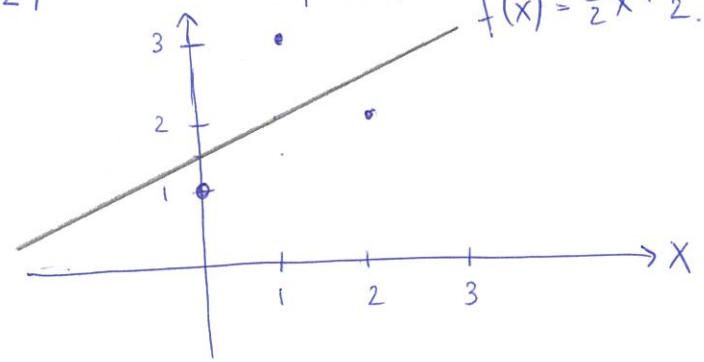
$$z = (y \circ f_1) f_1 + (y \circ f_2) f_2 = \frac{1}{3} \cdot 6 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot 1 \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 2 \\ \frac{5}{2} \end{pmatrix}.$$

This is an element in $\operatorname{im} A$ since it is a linear combination of f_1 and f_2.

iv) We want to solve $Ax = z$.

$$(A|z) \overset{(-2)}{\sim} \left(\begin{array}{cc|c} 0 & 1 & \frac{3}{2} \\ 1 & 1 & 2 \\ 2 & 1 & \frac{5}{2} \end{array}\right) \overset{(-1)}{\sim} \overset{(1)}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \\ 0 & -1 & -\frac{3}{2} \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{array}\right).$$

We get $a = \frac{1}{2}$ and $b = \frac{3}{2}$

The graph of f is the best fit for the three points.