Homework 6: Orthogonality & Gram-Schmidt algorithm

Deadline: 28th January, 2020

Gram-Schmidt algorithm (GSA): Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $U \subset \mathbb{R}^n$. The GSA constructs an orthonormal basis $F = (f_1, \dots, f_m)$ of U out of the basis B in the following way:

Step 1: Set $f_1 = \frac{1}{\|b_1\|} b_1$.

Step l $(2 \leq l \leq m)$: We have constructed orthonormal vectors $f_1, \dots, f_{l-1}$ in the steps before. Now set

$$w_l = b_l - (b_l \bullet f_1) f_1 - \cdots - (b_l \bullet f_{l-1}) f_{l-1} = b_l - \sum_{i=1}^{l-1} (b_l \bullet f_i) f_i$$

and define $f_l = \frac{1}{\|w_l\|} w_l$.

Exercise 1. (6 Points) We define the following vectors

$$b_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 4 \\ 0 \\ 5 \\ 8 \end{pmatrix}, \qquad b_3 = \begin{pmatrix} 8 \\ 1 \\ 5 \\ 6 \end{pmatrix}.$$

These form a basis $B = (b_1, b_2, b_3)$ of the subspace $U = \operatorname{span}\{b_1, b_2, b_3\} \subset \mathbb{R}^4$. Use the Gram-Schmidt algorithm to construct an orthonormal basis $F = (f_1, f_2, f_3)$ of U from B.

Exercise 2. (3+3 = 6 Points) Let $U \subset \mathbb{R}^n$ be a subspace.

i) Show that the orthogonal complement of U in $\mathbb{R}^n$, defined by

$$U^{\perp} = \{x \in \mathbb{R}^n \mid x \bullet u = 0 \,, \forall u \in U\} \,,$$

is a subspace.

ii) Determine a basis of $U^{\perp}$, where $U = \operatorname{span}\{b_1, b_2, b_3\}$ as in Exercise 1.

Exercise 3. (6 Points) Three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on one (non-vertical) line, if there exist $a, b \in \mathbb{R}$ with $ax_j + b = y_j$ for $j = 1, 2, 3$. In other words the linear system

$$\underbrace{\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{pmatrix}}_{=A} \begin{pmatrix} a \\ b \end{pmatrix} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{=y}$$

has a solution, i.e. $y \in \operatorname{im}(A)$.

i) Show that the points $(0, 1)$, $(1, 3)$ and $(2, 2)$ do <u>not</u> lie on one line.

From now on we assume that $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

ii) Calculate an orthonormal basis $F = (f_1, f_2)$ of $\operatorname{im}(A)$ by using the GSA for the columns of A. (Hint: The result becomes nicer if you set b_1 = second column of A and b_2 = first column of A.)

iii) Calculate $z = (y \bullet f_1) f_1 + (y \bullet f_2) f_2$ and show that $z \in \operatorname{im}(A)$.

iv) Solve the linear system $A \begin{pmatrix} a \\ b \end{pmatrix} = z$ and draw the graph of $f(x) = ax + b$ together with the three points in i). Can you interpret the connection between the graph and the points?