Homework 5: Bases & coordinates

Deadline: 14th January, 2020

Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $V \subset \mathbb{R}^n$. The **coordinate map** is defined by

$$c_B : \mathbb{R}^m \longrightarrow V\,,$$
$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \longmapsto \lambda_1 b_1 + \dots \lambda_m b_m\,.$$

The coordinate map c_B is bijective and therefore its inverse c_B^{-1} exists. In particular for each element $x \in V$ there exist unique $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, called the **coordinates of** x **in** B, with

$$x = \lambda_1 b_1 + \dots \lambda_m b_m\,.$$

The coordinates of $x \in V$ can calculated by

$$[x]_B = c_B^{-1}(x) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}\,,$$

which is called the **coordinate vector** of x .

Exercise 1. (6 Points) Let $U = \operatorname{span}\{u_1, u_2, u_3, u_4\} \in \mathbb{R}^4$, where

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of U.

ii) Calculate the coordinate vectors $[u_j]_B \in \mathbb{R}^m$ for $j = 1, 2, 3, 4$.

iii) For which values of $a \in \mathbb{R}$ does the vector $x = \begin{pmatrix} 3 \\ 3+a \\ 3+a \\ 2+2a \end{pmatrix}$ belong to U? For such a determine the coordinate vector $[x]_B$.

Exercise 2. (6 Points) For $t \in \mathbb{R}$ we define

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ t \end{pmatrix}, \quad v_3 = \begin{pmatrix} t \\ 4 \\ (t-2)^2 \end{pmatrix}$$

and set $V = \operatorname{span}\{v_1, v_2, v_3\}$. For each $t \in \mathbb{R}$ determine a basis of V and calculate its dimension.

The following exercise is intended to show the basic idea of 3D computer graphics, by showing how to get a 2-dimensional picture (to be shown on a 2-dimensional monitor) from an 3-dimensional object.

Exercise 3. (8 bonus points [1])

i) We define the corners of a cube with side length 18 in $\mathbb{R}^3$ by the following set

$$W = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3 \mid w_1, w_2, w_3 \in \{0, 18\} \right\} .$$

Make a drawing of a cube with side length 18 in $\mathbb{R}^3$, i.e. draw the 8 points in the set W and connect two points if they differ just by one entry.

ii) Show that $D = (d_1, d_2, d_3)$ is a basis of $\mathbb{R}^3$, where

$$d_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} .$$

iii) Calculate the coordinate vectors $[x]_D \in \mathbb{R}^3$ for all $x \in W$.

iv) Consider the following linear map

$$P : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 ,$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

Draw the points $P([x]_D) \in \mathbb{R}^2$ for $x \in W$ (This is now a picture in $\mathbb{R}^2$). Connect two points $P([x]_D)$ and $P([y]_D)$ in your picture if $x, y \in W$ just differ by one entry.

[1] The 8 bonus points are not part of the total amount of possible homework points. So if you already have 100% in your homework and you plan to always submit correct solutions, you do not need to do it. Otherwise, you can use it to improve your current homework points.