Homework 3: Linear maps & Geometry

Deadline: 26th November, 2019

Exercise 1. (4 Points) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that the following two statements are equivalent:

i) F is injective.

ii) The only solution to $F(x) = 0$ is $x = 0$.

(i.e. show that i) implies ii) and ii) implies i)).

Exercise 2. (2+2=4 Points) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

i) Calculate the matrices $[P_u]$ and $[\rho_u]$.

ii) Calculate the following vectors and draw them in one picture together with u,d and x

$$P_u(x), \quad \rho_u(x), \quad (P_u \circ P_d)(x), \quad \mathrm{rot}_{\frac{\pi}{2}}(x), \quad (P_u \circ \mathrm{rot}_{\frac{\pi}{2}})(x), \quad (\mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x), \quad (P_d \circ \mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x)\,.$$

Exercise 3. (2+2=4 Points) Show that for all $u \in \mathbb{R}^n$ with $u \neq 0$ the projection P_u and the reflection ρ_u satisfy for all $x \in \mathbb{R}^n$ the following two properties:

i) $P_u(P_u(x)) = P_u(x)$.

ii) $\rho_u(\rho_u(x)) = x$.

Exercise 4. (2+2=4 Points)

i) Show that for $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$ with $u \neq 0$ the matrix of P_u is given by

$$[P_u] = \frac{1}{u \bullet u} u u^T \in \mathbb{R}^{n \times n}\,,$$

where $u^T = (u_1\, u_2\, \dots\, u_n) \in \mathbb{R}^{1 \times n}$. Use this to give an expression for $[\rho_u]$.

ii) Show that a vector $x \in \mathbb{R}^n$ satisfies $[P_u]x = 0$ if and only if $u \bullet x = 0$. Give a geometric explanation of this result, and draw a picture illustrating the situation for $n = 2$.

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