Homework 2: Matrices, Vectors & Linear Maps

Deadline: 5th November, 2019

Exercise 1. (4 Points) Let $a, b, c, d \in \mathbb{R}$ with $ad - bc \neq 0$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that $\mathrm{rk}(A) = 2$.

Exercise 2. (2+2 = 4 Points) Show that the multiplication with a matrix gives rise to a linear map, i.e. prove that the following identities hold for all $A \in \mathbb{R}^{m \times n}$, $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$:

i) $A(x + y) = Ax + Ay$,

ii) $A(\lambda x) = \lambda(Ax)$.

Exercise 3. (4 Points) Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be a polynomial of degree 3 with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For this polynomial p we define the vector v_p by

$$v_p = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^4 .$$

Find a matrix $D \in \mathbb{R}^{4 \times 4}$, such that $v_{p'} = D v_p$, where p' denotes the derivative of the polynomial p with respect to x. What is the rank of D?

Exercise 4. (1+1+1+1=4 Points) Which of the following functions are linear maps?

$$\begin{aligned} f_1 : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto e^x , \end{aligned}$$

$$\begin{aligned} f_2 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \\ x_1 - x_2 \end{pmatrix} , \end{aligned}$$

$$\begin{aligned} f_3 : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto 3x + 2 , \end{aligned}$$

$$\begin{aligned} f_4 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + 2x_2 \\ x_1 x_2 \end{pmatrix} . \end{aligned}$$

Exercise 5. (4 Points) Show that there exist a <u>unique</u> linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} , \qquad T \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} .$$

What is the value of $T(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of T.