Linear algebra I

Final exam

1) $A = \begin{pmatrix} 0 & 1 & -2 & 3 \\ 1 & -2 & 3 & -4 \\ -2 & 3 & -4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

i) • AB is not defined.

• $BA = \begin{pmatrix} -1 & 3 & -5 & 7 \\ 0 & 2 & -4 & 6 \\ -3 & 6 & -9 & 12 \end{pmatrix}$.

ii) A is not invertible because it is not a square matrix.

$$(B \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right).$$

B is invertible with inverse $B^{-1} = \begin{pmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$.

iii) Since B is invertible we have

$$\operatorname{im}(B) = \mathbb{R}^3 \quad \text{and} \quad \operatorname{Ker}(B) = \{0\}.$$

2) $U = \text{span}\{u_1, u_2, u_3\}$

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

i) To determine a basis for U we calculate the rref of $\begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & -3 & -1 \end{pmatrix} \overset{(-1)}{\sim} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{pmatrix} \overset{(3)}{\sim} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

pivot.

The first two columns contain pivot elements and therefore $B = (b_1, b_2)$ with $b_1 = u_1$, $b_2 = u_2$ is a basis of U and $\dim U = 2$.

ii) From the above rref we can read off:

$$[u_1]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [u_2]_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [u_3]_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

iii) To calculate a basis for $U^\perp$ we need to find all $x \in \mathbb{R}^3$ with $x \cdot b_1 = x \cdot b_2 = 0$. This is equivalent to finding the solutions of $\begin{pmatrix} - b_1 - \\ - b_2 - \end{pmatrix} x = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix} x = 0$.

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right).$$ We have the solutions

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \begin{aligned} x_1 &= +3t \\ x_2 &= -t \\ x_3 &= t \end{aligned} \qquad t \in \mathbb{R}$$

$$\Rightarrow x = t \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \left(\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right) \text{ is a basis for } U^\perp.$$

$$U^\perp = \text{span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

iv) We define
$$G : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \longmapsto \lambda_1 b_1 + \lambda_2 b_2 .$$

This is the coordinate map C_B for which we know that $\mathrm{im}(G) = U$ and $\ker G = \{0\}$.

v) Let $[G]$ be the matrix of G in iv), i.e $[G] = \begin{pmatrix} \vdots & \vdots \\ b_1 & b_2 \\ \vdots & \vdots \end{pmatrix}$
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -3 \end{pmatrix}.$$

Then we know that $\mathrm{im}([G]) = U$. On the other hand we have $U^\perp = (\mathrm{im}([G]))^\perp = \ker [G]^T$.

Therefore if we define
$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto [G]^T x = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \end{pmatrix} x$$

then we have $\ker H = U^\perp$, since $[H] = [G]^T$.

This question was removed and was <u>not</u> part of the exam.
It was: "Find a linear map $H : \mathbb{R}^3 \to \mathbb{R}^2$ with $\ker H = U^\perp$.

3)

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = x_1 x_2 \right\}$

This is <u>not</u> a subspace, because $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \in U_1$,

but $\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \notin U_1$, since $4 + 4 \neq 4 \cdot 4$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 = x_1 + x_2 \right\}$.

Since $2x_1 = x_1 + x_2 \Leftrightarrow x_1 = x_2$ we have $U_2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

and therefore U_2 is a subspace.

iii) $U_3 = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \cup \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

We have $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \in U_3$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \in U_3$, but $\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

is not an element in U_3, since it is not a multiple of $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

<u>or</u> a multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Therefore U_3 is <u>not</u> a subspace.

iv) $U_4 = U_2 \cap U_3$.

Since $U_2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}$ we have $U_4 = U_2$.

Therefore U_2 is also a subspace.

This question was <u>not</u> part of the exam.

It was "check if $U_4 = U_2 \cap U_3$ is a subspace".

4)

i) We first notice that $b_1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$, $b_2 = \begin{pmatrix}2\\3\\1\end{pmatrix}$ form a basis of $\operatorname{im}(T)$, since they are linearly independent.

Using the Gram-Schmidt algorithm we get:

$$f_1 = \frac{1}{\|b_1\|} b_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\\-1\end{pmatrix}.$$

$$w_2 = b_2 - (b_2 \cdot f_1) f_1 = \begin{pmatrix}2\\3\\1\end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\\6\\3\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$f_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{\sqrt{6}} \begin{pmatrix}1\\2\\1\end{pmatrix}.$$

Then $F = (f_1, f_2)$ is an orthonormal basis of $\operatorname{im}(T)$.

ii) (Since $f_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$ is a basis vektor, $v = \begin{pmatrix}1\\t\\1\end{pmatrix}$ can just be an element of $\operatorname{im}(T)$ if $t=2$.)

Alternative: We try to solve $T(x) = v$:

$$\left(\begin{array}{cc|c}1 & 2 & 1\\0 & 3 & t\\-1 & 1 & 1\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 2 & 1\\0 & 3 & t\\0 & 3 & 2\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 2 & 1\\0 & 3 & t\\0 & 0 & 2-t\end{array}\right)$$

$\Rightarrow$ This just has a solution for $t=2$.

For $t=2$ we have $v = \begin{pmatrix}1\\2\\1\end{pmatrix}$ and since $v = \sqrt{6} f_2 = \frac{\sqrt{6}}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$

we get $[v]_F = \begin{pmatrix}0\\\sqrt{6}\end{pmatrix}$.

iii) A $w \in \mathbb{R}^3$ with $[w]_F = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is given by

$$1 \cdot f_1 + 2 \cdot f_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{2}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{6}} \\ \frac{4}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{3}+2}{\sqrt{6}} \\ \frac{4}{\sqrt{6}} \\ \frac{-\sqrt{3}+2}{\sqrt{6}} \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{3}+2 \\ 4 \\ -\sqrt{3}+2 \end{pmatrix}.$$

iv) To find a $x \in \mathbb{R}^2$ such that $\|T(x) - \underset{\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}{\underset{\shortparallel}{b}}\|$ is minimal, we need to solve

$$A^T A x = A^T b, \text{ with } A = [T] = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix}.$$

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 14 \end{pmatrix}.$$

$$A^T b = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}.$$

Need to solve $\begin{pmatrix} 2 & 1 \\ 1 & 14 \end{pmatrix} x = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 2 & 1 & -3 \\ 1 & 14 & 12 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & -27 & -27 \\ 1 & 14 & 12 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 14 & 12 \\ 0 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right).$$ We get the solution $x = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$,

i.e $\left\|T\begin{pmatrix} -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right\|$ is minimal.