Linear Algebra I - Final exam

Nagoya University, G30 Program

Fall 2019

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1) (12 Points) Let $A = \begin{pmatrix} 0 & 1 & -2 & 3 \\ 1 & -2 & 3 & -4 \\ -2 & 3 & -4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

i) Compute the products AB and BA, or explain why they are not defined.

ii) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.

iii) Calculate $\mathrm{im}(B)$ and $\ker(B)$.

2) (14 Points) We define the subspace $U = \mathrm{span}\{u_1, u_2, u_3\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$ and $[u_3]_B$, where B is the basis you determined in i).

iii) Determine a basis for $U^\perp$.

iv) Find a linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with $\ker(G) = \{0\}$ and $\mathrm{im}(G) = U$.

3) (10 Points) Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = x_1 x_2 \right\}$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 = x_1 + x_2 \right\}$.

iii) $U_3 = \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \bigcup \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

(Friendly reminder: $\cup$ is the union of two sets)

4) (14 Points) We define the following linear map

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

i) Calculate an orthonormal basis $F = (f_1, \dots, f_r)$ for $\mathrm{im}(T)$.

ii) Check for which $t \in \mathbb{R}$ the vector $v = \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}$ is an element in $\mathrm{im}(T)$. Determine the coordinate vector $[v]_F$ in this case.

iii) Find a $w \in \mathbb{R}^3$ with $[w]_F = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

iv) Find a $x \in \mathbb{R}^2$ such that $\|T(x) - b\|$ is minimal, where $b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

(In ii) and iii) the F is the basis of $\mathrm{im}(T)$ you calculated in i)).

Do not turn this page before told to do so!

- This exam consists of four problems, with a total score of 50 points.
- Time: 90 minutes (10:30 - 12:00).
- All solutions should be clear answers to the questions asked. Justify your answers!
- According to Nagoya University student discipline rules, cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the term.
- You are not allowed to have your phone near you. In particular, it is not allowed to have your phone in your pants pocket.
- Materials allowed: Writing material & non-programmable calculator.
- Do not use your own paper. Writing paper will be provided. Please use both sides of the paper.
- It is possible to leave the room during the exam to go to the toilet (Just one person at a time).
- You can hand in your exam before the time is over. If you do so, please leave the room quietly!
- Do not forget to write your name on each piece of paper you hand in!

Good luck!