Formalization of multiple Eisenstein series

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Part 4: Derivations & sl_2-action on $\mathcal{G}^f$

Recall Part 1: bi-setup $A^{bi} = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \,\middle|\, k \geq 1, d \geq 0 \right\}$

$$aw * bv = a(w * bv) + b(aw * v) + (a \diamond b)(w * v), \qquad \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 \\ d_1+d_2 \end{bmatrix}$$

Definition The algebra of FMES is defined by

$$\mathcal{G}^f = (\mathbb{Q}\langle A^{bi}\rangle, *) \big/ \langle \sigma(w) - w \mid w \in \mathbb{Q}\langle A^{bi}\rangle\rangle.$$

- We write $G^f\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix}$ for the class of $\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix}$.

- We set $G^f(k_1, \dots, k_r) := G^f\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix}$.

Set $\mathcal{G}^{f,0} := \langle G^f(k_1,\dots,k_r) \mid k_1,\dots,k_r \geq 1, r \geq 0 \rangle_{\mathbb{Q}}$

Notice: $\mathcal{G}^{f,0}$ is a subalgebra

<u>Conjecture</u>: $\mathcal{G}^f = \mathcal{G}^{f,0}$.

(work in progress
B.-Brindle-van-Ittersum)

Example: $G^f\binom{2,1}{1,0} = G^f(3,1) - G^f(2,1,1) + G^f(2,2)$

4.1 CMES & depth one

In depth one the FMES satisfy the same relations as classical Eisenstein series and their derivatives.

More precisely: For an $\mathbb{Q}$-algebra R we call an algebra homomorphism $\mathcal{G}^f \to R$ a realization of $\mathcal{G}^f$ in R.

Then we have:

<u>Theorem</u> (B.-Burmester, 2022) There exists a realization

$$G: \mathcal{G}^f \longrightarrow \mathbb{Q}[[q]]$$

with $$G^f(k) \longmapsto G(k) := -\frac{B_k}{2k!} + g(k) \quad (k \geq 2)$$

More generally: For $k > d \geq 0$

$$G^f\binom{k}{d} \longmapsto \frac{(k-d-1)!}{(k-1)!}\left(q\frac{d}{dq}\right)^d G(k-d),$$

i.e. $$G^f\binom{k-1}{1} \longmapsto \frac{1}{k-2}\, q\frac{d}{dq} G(k-2)$$

We expect that $\mathcal{G}$ is injective. In particular, in depth one the G^f satisfy the same algebraic relations as Eisenstein series, which is known:

Theorem (B.-Kühn-Matthes, 2020) i) For even $k \geq 4$

$$\frac{k+1}{2} G^f(k) = G^f\binom{k-1}{1} + \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \geq 2 \text{ even}}} G^f(k_1)\, G^f(k_2)$$

ii) For all even $k \geq 6$ we have

$$\frac{(k+1)(k-1)(k-6)}{12} G^f(k) = \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \geq 4 \text{ even}}} (k_1-1)(k_2-1)\, G^f(k_1) G^f(k_2)$$

In particular, $G^f(k) \in \underbrace{\mathbb{Q}[G^f(4), G^f(6)]}_{\text{formal modular forms.}}^{}$ $k \geq 8$ even

$\mathbin{/\!/}$ formal modular forms.

bi-version of projection p seen in part 3:

Theorem (B.-Matthes-v.Ittersum, 2021)
unpublished note

i) There exists a surjective algebra homomorphism

$$\pi : \mathcal{G}^f \longrightarrow \mathcal{Z}^f .$$

"projection to the constant term"

ii) The kernel of π is the ideal generated by $G^f(1)$ and all elements which are not of the form

$$G^f\binom{1, \dots, 1, k_1, \dots, k_r}{d_1, \dots, d_s, 0, \dots, 0} \qquad s, r \geq 0 .$$

iii) $\pi(\mathcal{G}^{f,0}) = \mathcal{Z}^f .$

4.2 Quasimodular forms & sl_2-algebras

Let $M = \mathbb{Q}[G(4), G(6)] \subset \widetilde{M} = \mathbb{Q}[G(2), G(4), G(6)]$ be the rings of modular & quasimodular forms (with rat. coefficients)

Denote by M_k, $\widetilde{M}_k$ the weight k parts.

Write $D = q\frac{d}{dq}$.

Then we have the following well known facts:

Proposition i) We have

$$DG(2) = 5G(4) - 2G(2)^2,$$

$$DG(4) = 8G(6) - 14G(2)G(4),$$

$$DG(6) = \frac{120}{7}G(4)^2 - 12G(2)G(6)$$

and therefore $q\frac{d}{dq}\widetilde{M}_k \subset \widetilde{M}_{k+2}$.

ii) Any $f \in \widetilde{M}_k$ can be written uniquely as

$$f = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} f_j \, G(2)^j \quad \text{with } f_j \in M_{k-2j}.$$

On $\widetilde{M}$ we can define two more derivations defined on the weight graded parts by

$$W: \widetilde{M}_k \longrightarrow \widetilde{M}_k \qquad \text{weight operator}$$
$$f \longmapsto kf$$

$$\delta: \widetilde{M}_k \longrightarrow \widetilde{M}_{k-2} \qquad \text{"Derivative with respect to } G(2)\text{"}$$

determined by $\delta(G(2)) = -\frac{1}{2}$, $\delta(G(4)) = \delta(G(6)) = 0$.

Notice: $M = \ker \delta$.

Lie algebra sl_2: • 3-dimensional Lie algebra.

• Matrix representation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a+b=0$

Basis: $\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sim H}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\sim X}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\sim Y}$

• A triple of operators (X, H, Y) is called <u>sl_2-triple</u> if

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H.$$

• If these operators act on an algebra A by derivations the A is called an <u>sl_2-algebra</u>.

<u>Proposition</u>: (δ, W, D) is an sl_2-triple $\Rightarrow \widetilde{M}$ sl_2-algebra

4.3 Formal quasimodular forms & Derivations on $\mathcal{G}^f$

<u>Theorem</u> (B. - v. Ittersum, 2023)

i) There exist derivations (δ^f, W^f, D^f) which make $\mathcal{G}^f$ into a $\mathfrak{sl}_2$-algebra.

ii) $\mathcal{G}^{f,0}$ is a $\mathfrak{sl}_2$-subalgebra.

<u>Definition</u> i) We define the formal quasimodular forms $\widetilde{\mathcal{M}}^f$ as the smallest $\mathfrak{sl}_2$-subalgebra of $\mathcal{G}^f$ containing $G^f(2)$.

ii) We define the algebra of formal modular forms by $\mathcal{M}^f := \ker \delta^f_{\widetilde{M}}$.

<u>Theorem</u> (B. - v. Ittersum) i) $\widetilde{\mathcal{M}}^f \cong \widetilde{\mathcal{M}}$ as $\mathfrak{sl}_2$-algebras

ii) $\mathcal{M}^f \cong \mathcal{M}$.

The derivations are given as follows:

$$W^f\left(G^f\binom{k_1 \ldots k_r}{d_1 \ldots d_r}\right) = (k_1 + \ldots + k_r + d_1 + \ldots + d_r)\, G^f\binom{k_1 \ldots k_r}{d_1 \ldots d_r}$$

$$D^f\left(G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r}\right) = \sum_{j=1}^{r} k_j \, G^f\binom{k_1,\dots,k_j+1,\dots,k_r}{d_1,\dots,d_j+1,\dots,d_r}$$

<u>Proposition</u> (B.–Burmester) G is differential alg. hom.

$\delta^f(\dots)$ = sum of 5 derivations on $(\mathbb{Q}\langle A\rangle, *)$, which (by magic) commute with σ

Remark: • We define these derivations on $(\mathbb{Q}\langle A\rangle, *)$ and show that the commute with σ

They seem to be unique with that property!

• There seem to be also derivations of negative odd weight.

• On $\mathcal{G}^{f,0}$ we have: (view G^f as a map $\mathfrak{h}^1 \to \mathcal{E}_0^f$, $z_k \mapsto G^f(u)$)

$w \in \mathfrak{h}^1$

$$D^f(G^f(w)) = G^f(z_2 * w - z_2 ш w)$$

$$\delta^f\left(G^f(k_1,\dots,k_r)\right) = -\frac{1}{2}\,\mathbb{1}_{k_1=2}\,G^f(k_2,\dots,k_r)$$

$$+\frac{1}{4}\,\mathbb{1}_{k_1=k_2=1}\,G^f(k_3,\dots,k_r) \qquad \mathbb{1}_P=\begin{cases}0 & P \text{ false}\\ 1 & \text{else}\end{cases}$$

$$+\frac{1}{2}\sum_{j=1}^{r-1}\mathbb{1}_{\substack{k_j=1\\ k_{j+1}>1}}\,G^f(k_1,\dots,k_{j-1},k_{j+1}-1,\dots,k_r)$$

$$-\frac{1}{2}\sum_{j=2}^{r}\mathbb{1}_{\substack{k_j=1\\ k_{j-1}>1}}\,G^f(k_1,\dots,k_{j-1}-1,k_{j+1},\dots,k_r)$$

Summary:

$$A=\left\{\begin{bmatrix}k\\ d\end{bmatrix} \;\middle|\; \begin{matrix}k\geq 1\\ d\geq 0\end{matrix}\right\} \qquad B=\{b_k \mid k\geq 0\}$$

$$G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \quad \underbrace{\mathbb{Q}\langle A\rangle/S}_{\text{bi}} \cong \underbrace{\mathbb{Q}\langle B\rangle^0/T}_{\text{balanced}}$$

(∗-reg) Formal MZV

Formal MES

CMES $\mathcal{G}$

$$\begin{tikzcd}
 & \mathcal{Z}^f \arrow[dl, "\beta"', "\mathbb{Q}\text{-realization}"] \arrow[dd, "\varphi_u^*"] & & \mathcal{G}^f \arrow[ll, "\pi^f\ \text{``}q=0\text{''}"'] \arrow[dl, "\text{[BB]}"'] \arrow[dd, "G^*\ (\text{conj.})"] \arrow[loop above, "\mathfrak{sl}_2\ D\ \delta\ W"] \\
\mathbb{Q} & & \mathcal{G}_{\mathbb{Q}} \arrow[ll, "\pi\ (q=0)"'] \arrow[dl, "\text{``}q\to 1\text{''}"] & \\
 & \mathcal{Z} & & \mathcal{G} \arrow[ll, "\pi\ (q=0)"]
\end{tikzcd}$$

$\mathcal{G}_{\mathbb{Q}} \subset \mathbb{Q}[[q]]$, $\quad \mathcal{Z}\subset\mathbb{R}$, $\quad \mathcal{G}\subset\mathcal{O}(\mathbb{H})$

$\mathcal{G}^f = \langle G^f(k_1,\dots,k_r) \mid k_j\geq 1\rangle$

$\cup$

$\widetilde{\mathcal{M}}^f =$ smallest $\mathfrak{sl}_2$-subals. containing

$\cup$

$\widetilde{\mathcal{M}} = \ker\delta$

$\mathcal{G} \supset \langle \mathbb{G}_{k_1,\dots,k_r}(\tau) \mid k_j\geq 2\rangle_{\mathbb{Q}}$

$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \displaystyle\sum_{\lambda_1>\dots>\lambda_r>0}\frac{1}{\lambda_1\cdots}$