Formalization of multiple Eisenstein series

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Henrik Bachmann, Annika Burmester

Part 1: (Formal) Multiple Eisenstein Series.
(B.) History / Motivation, Definition & some results
(The bi setup)

Part 2: The balanced setup, balanced $\Leftrightarrow$ bi.
(Burmester)

Part 3: An analogue of Racinet's approach to formal multiple zeta values.
(Burmester)

Part 4: Derivations and $\mathfrak{sl}_2$-action on $\mathfrak{g}^f$.
(B.)

Part 1: (Formal) Multiple Eisenstein series ((F)MES)

1.1 Multiple zeta values (MZV)

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ we define the MZV

$\mathbf{k} = (k_1, \ldots, k_r)$

$$\zeta(\mathbf{k}) := \zeta(k_1, \ldots, k_r) := \sum_{m_1 > \ldots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}.$$

MZV satisfy various relations, e.g. $\zeta(2,1) = \zeta(3)$, and
$\zeta(3,3) - \frac{1}{2}\zeta(4,2) = \frac{1}{6}\zeta(6)$

Some can be computed explicitly, e.g. $\zeta(\underbrace{2k, \ldots, 2k}_{n}) \in \pi^{2kn}\mathbb{Q}$.

weight: $\mathrm{wt}(\mathbf{k}) = k_1 + \ldots + k_r$, depth: $\mathrm{dep}(\mathbf{k}) = r$

Hoffmans algebraic setup:

word := monic monomials

$$\mathfrak{h}^0 = \mathbb{Q} + x\mathfrak{h}y \subset \mathfrak{h}^1 = \mathbb{Q} + \mathfrak{h}y \subset \mathfrak{h} = \mathbb{Q}\langle x,y\rangle$$

$$\langle z_{\mathbf{k}} \mid \mathbf{k} \text{ admissible}\rangle \subset \langle z_{k_1}\cdots z_{k_r} \mid r \geq 0, k_j \geq 1\rangle_{\mathbb{Q}} \qquad z_k = x^{k-1}y$$

$$z_{k_1}\cdots z_{k_r} = z_{\mathbf{k}} \qquad \mathbf{k} = (k_1,\dots,k_r)$$

$k_1 \geq 2$

On $\mathfrak{h}$ we can define the shuffle product:

$$aw \sqcup\!\sqcup\, bv = a(w \sqcup\!\sqcup\, bv) + b(aw \sqcup\!\sqcup\, v)$$

$a,b \in \{x,y\}$

w,v word in $\mathfrak{h}$

$w \sqcup\!\sqcup\, 1 = 1 \sqcup\!\sqcup\, w = w$

empty word

and on $\mathfrak{h}^1$ the stuffle product

$$z_i w * z_j v = z_i(w * z_j v) + z_j(z_i w * v) + z_{i+j}(w * v).$$

(extending both bilinearly)

$i,j \geq 1$, w,v word in $\mathfrak{h}^1$

$w * 1 = 1 * w = w$.

Notice: $\mathfrak{h}^0, \mathfrak{h}^1$ are closed under $\sqcup\!\sqcup$ and $\mathfrak{h}^0$ under $*$.

Obtain commutative $\mathbb{Q}$-algebras

$$\mathfrak{h}^0_{\sqcup\!\sqcup} \subset \mathfrak{h}^1_{\sqcup\!\sqcup} \subset \mathfrak{h}_{\sqcup\!\sqcup}$$

$$\mathfrak{h}^0_{*} \subset \mathfrak{h}^1_{*}$$

by Miya

Fact: For $\bullet \in \{*, ш\}$ the $\mathbb{Q}$-linear map

$$\zeta : \mathfrak{h}^0_\bullet \longrightarrow \mathbb{R}$$
$$z_{\mathbf{k}} \longmapsto \zeta(\mathbf{k})$$

is an algebra homomorphism.

(Proof: For $*$ use definition as iterated sum + induction, $ш$ use Polylogarithm & iterated integrals)

$\mathcal{Z} := \operatorname{im} \zeta$: $\mathbb{Q}$-alg. of MZV

<u>Regularization:</u> For $\bullet \in \{*, ш\}$ there exist unique algebra homomorphisms

$$\zeta^\bullet : \mathfrak{h}^1_\bullet \longrightarrow \mathcal{Z}$$

with $\zeta^\bullet(z_1) = \zeta^\bullet(y) = 0$ and $\zeta^\bullet_{|\mathfrak{h}^0} = \zeta$.

Let $EDS^\bullet$ be the ideal in $\mathfrak{h}^1_\bullet$ generated by z_1 and $w * v - w ш v$ with $w \in \mathfrak{h}^0, v \in \mathfrak{h}^1$.

<u>Conjecture:</u> $\operatorname{Ker} \zeta^\bullet = EDS^\bullet$

Formal MZV: $\mathcal{Z}^f := \mathfrak{h}^1_* / EDS^* \overset{\substack{\text{Conj} \\ \Downarrow}}{\cong} \mathcal{Z}$

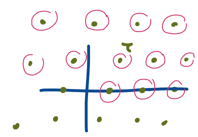
$\zeta^f(\mathbf{k})$: class of $z_{\mathbf{k}}$.

1.2 MES and their history

There are various (partly conjectured) connections of MZV and modular forms. $k \geq 2$

Classical Eisenstein series: $k \geq 4$ even

$$\frac{1}{2}\sum_{\lambda \in \mathbb{Z}\tau+\mathbb{Z}\setminus 0} \frac{1}{\lambda^k}$$



$$= \mathbb{G}(k) = \mathbb{G}(k;\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\frac{n^{k-1}q^n}{1-q^n} = g(k)$$

$$\overset{k\geq 3}{=} \sum_{\lambda \succ 0} \frac{1}{\lambda^k}$$

$m\tau+n \succ 0 \iff m>0$ or $m=0, n>0$

ring of Modular forms: $\mathcal{M} = \mathbb{C}[\mathbb{G}(4), \mathbb{G}(6)]$.

$\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{im}(\tau) > 0\}$.

$\lambda_1 \succ \lambda_2 :\iff \lambda_1 - \lambda_2 \succ 0$

2006 Gangl-Kaneko-Zagier:

Double Eisenstein series: $\displaystyle\sum_{\lambda_1 \succ \lambda_2 \succ 0} \frac{1}{\lambda_1^{k_1}\lambda_2^{k_2}} = \zeta(k_1,k_2) + \sum_{n\geq 1} a_n q^n$

~ Tanaka: Triple Eisenstein series

2012 B. (master thesis) Multiple Eisenstein series

$k_1, \dots, k_r \geq 2$, $\mathbb{k} = (k_1, \dots, k_r)$

$$\mathbb{G}(\mathbb{k}) = \mathbb{G}(\mathbb{k};\tau) = \sum_{\lambda_1 \succ \dots \succ \lambda_r \succ 0} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}} \in \mathcal{O}(\mathbb{H})$$

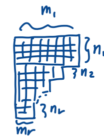
ring of hol. fct. in $\mathbb{H}$

<u>Thm</u> (B.) $\mathbb{G}$ can be written as an explicit $\mathbb{Z}[\pi i]$-lin. comb of

$$g(\mathbb{k}) = g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1-1}}{(k_1-1)!} \cdots \frac{n_r^{k_r-1}}{(k_r-1)!} q^{m_1 n_1 + \dots + m_r n_r}$$

$$= \sum_{N\geq 1} \left(\sum_{\substack{\text{partitions}\\ \text{of } N \text{ with}\\ r \text{ different}\\ \text{parts}}} \frac{n_1^{k_1-1}}{(k_1-1)!} \cdots \frac{n_r^{k_r-1}}{(k_r-1)!} \right) q^N$$

※ $\square = N$



$$N = \overbrace{m_1+\dots+m_1}^{n_1 \leftarrow \text{multiplicities}} + \dots + \overbrace{m_r+\dots+m_r}^{n_r}$$

m_1: part

More precisely: There exist $\alpha^{\mathbb{k}}_{\mathbb{a},\mathbb{b}} \in \mathbb{Z}$ with

$$\mathbb{G}(\mathbb{k}) = \mathfrak{g}(\mathbb{k}) + \sum_{\substack{\mathrm{dep}(\mathbb{a})+\mathrm{dep}(\mathbb{b}) = \mathrm{dep}(\mathbb{k})\\ \mathrm{wt}(\mathbb{a})+\mathrm{wt}(\mathbb{b}) = \mathrm{wt}(\mathbb{k})\\ 1 < \mathrm{dep}(\mathbb{a}),\mathrm{dep}(\mathbb{b}) < \mathrm{dep}(\mathbb{k})}} \alpha^{\mathbb{k}}_{\mathbb{a},\mathbb{b}}\, \mathfrak{g}(\mathbb{a})\, \hat{g}(\mathbb{b}) + \hat{g}(\mathbb{k}),$$

where $\hat{g}(\mathbb{k}) = (-2\pi i)^{\mathrm{wt}(\mathbb{k})} g(\mathbb{k})$.

$\mathbb{G}(3,2) = \mathfrak{g}(3,2) + 3\,\hat{g}(2)\,\mathfrak{g}(3) + 2\,\hat{g}(3)\,\mathfrak{g}(2) + \hat{g}(3,2)$

Example: • $\mathbb{G}(k) = \mathfrak{g}(k) + \hat{g}(k)$

• $\mathbb{G}(3,2) = \mathfrak{g}(3,2) + 2\,\mathfrak{g}(2)\hat{g}(3) + 3\,\mathfrak{g}(3)\hat{g}(2) + \hat{g}(3,2)$

Question: • Good definition of $\mathbb{G}(\mathbb{k})$ for any $\mathbb{k} \in \mathbb{Z}^r_{\geq 1}$?

In other words: If we view $\mathbb{G}$ as a $\mathbb{Q}$-linear map

$$\mathbb{G} : \mathfrak{h}^2 \longrightarrow \mathcal{O}(\mathbb{H})$$

$$\mathfrak{h}^2 = \langle z_{k_1}\cdots z_{k_r} \mid k_1,\dots,k_r \geq 2 \rangle_{\mathbb{Q}}$$

($\mathfrak{h}^2$ is a subalgebra of $\mathfrak{h}^1_*$ and $\mathbb{G}$ is alg. hom.)

can we extend it to alg. hom. $\mathbb{G}^\bullet : \mathfrak{h}^1_\bullet \to \mathcal{O}(\mathbb{H})$ with $\mathbb{G}^\bullet|_{\mathfrak{h}^2} = \mathbb{G}$ and $\mathbb{G}^\bullet(\mathbb{k})|_{q=0} = \mathfrak{g}^\bullet(\mathbb{k})$?

2014 (published 2017) B.-Tasaka: Shuffle regularized MES $\mathbb{G}^{ш} : \mathfrak{h}^1_{ш} \to \mathcal{O}(\mathbb{H})$

2012 – 2015 B. (PhD thesis)

Study of the q-series $g(K)$

"What relations do they satisfy?"

comes from bi-moulds

Introduction of bi versions of g (Motivated by the interpretation as sum over partitions)

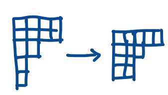
For $k_1,\dots,k_r \geq 1$, $d_1,\dots,d_r \geq 0$

$$g\begin{pmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{pmatrix} = \sum_{\substack{m_1>\dots>m_r>0 \\ n_1,\dots,n_r>0}} \prod_{j=1}^{r} \frac{n_j^{k_j-1}}{(k_j-1)!} m_j^{d_j} q^{m_j n_j}$$

(Note: $g(k_1,\dots,k_r) = g\begin{pmatrix} k_1,\dots,k_r \\ 0,\dots,0 \end{pmatrix}$)

$$= \sum_{N>0} \Big(\sum \text{pol. in parts \& mult.} \Big) q^N$$

- Interpretation as sum over partitions:

  g is invariant under conjugation ex: $r=3$ 

  Example: $g\begin{pmatrix} 3,1 \\ 1,0 \end{pmatrix} = \frac{1}{2} g\begin{pmatrix} 1,2 \\ 0,2 \end{pmatrix} + \frac{1}{2} g\begin{pmatrix} 2,1 \\ 0,2 \end{pmatrix}$

  "swap invariant"

- g satisfy an analogue of stuffle product "q-stuffle"

Example: $g\begin{pmatrix} 1 \\ 2 \end{pmatrix} g\begin{pmatrix} 3 \\ 4 \end{pmatrix} = g\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + g\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} + g\begin{pmatrix} 4 \\ 6 \end{pmatrix} - \frac{1}{2} g\begin{pmatrix} 3 \\ 6 \end{pmatrix} + \frac{1}{12} g\begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$$\mathcal{Z}_q := \left\langle g\begin{pmatrix} k_1 \dots k_r \\ d_1,\dots,d_r \end{pmatrix} \;\middle|\; \begin{matrix} k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 0 \end{matrix} \right\rangle_{\mathbb{Q}}$$

Conjectures / Observation (B., ~2015)

① The $\mathbb{Q}$-lin. rel. among $\mathbb{G}(k)$ are exactly those of $g(k)$ modulo lower weight. (when restricting to $k_j \geq 2$)

Example:

$$g(3,3) - \tfrac{1}{2} g(4,2) - \tfrac{1}{6} g(6) + \tfrac{1}{24} g(4) - \tfrac{1}{180} g(2) = 0$$

$$\mathbb{G}(3,3) - \tfrac{1}{2}\mathbb{G}(4,2) - \tfrac{1}{6}\mathbb{G}(6) = 0$$

$\mathbb{G}(3,3) = \zeta(3,3) + \zeta(3)\hat{g}(3) - 6\zeta(4)\hat{g}(2) + \hat{g}(3,3)$

$-\tfrac{1}{2}\mathbb{G}(4,2) = -\tfrac{1}{2}\zeta(4,2) - \zeta(2)g(4) - \zeta(3)\hat{g}(3) - 2\zeta(4)\hat{g}(2) - \tfrac{1}{2}\hat{g}(4,2)$

$-\tfrac{1}{6}\mathbb{G}(6) = -\tfrac{1}{6}\zeta(6) - \tfrac{1}{6}\hat{g}(6)$

$$\frac{-\zeta(2)}{(-2\pi i)^2} = \frac{1}{24} \qquad \frac{-6\zeta(4) - 2\zeta(4)}{(-2\pi i)^4} = -\frac{1}{180}$$

② All $\mathbb{Q}$-linear relations among (bi) g are a consequence of
- swap invariance
- g-stuffle product

③ $\mathcal{Z}_q = \langle g(k) \mid k \in \mathbb{Z}_{\geq 1}^r, r \geq 0 \rangle_{\mathbb{Q}}$

① + ②: Motivation for formal MES

swap inv. symbols satisfying g-stuffle mod lower weight

Other results.

B.: stuffle reg. MES: $\mathbb{G}^*: \mathfrak{h}^1_* \to \mathcal{O}(\mathbb{H})$.

B.-Kühn: • $g(k)$ are q-analogues of MZV (qMZV)

$$\lim_{q \to 1} (1-q)^{\mathrm{wt}(k)} g(k) = \zeta(k)$$

(B.-van-Ittersum 2022: true for all $g\binom{k_1 \ldots k_r}{d_1 \ldots d_r}$)

- $\mathcal{Z}_q$ contains a lot of different models for qMZV
- Dimension conjectures

2020 Formal MES $\mathcal{G}^f = \dfrac{(\mathbb{Q}\langle A^{bi}\rangle, *)}{\langle \sigma(w)-w \mid w \in \mathbb{Q}\langle A^{bi}\rangle\rangle}$ q-stuffle mod lower weight; swap invariance

2020 B.-Matthes-Kühn: Realizations for Formal double Eisenstein series

2020 Burmester (PhD thesis):
- (+B.) Combinatorial MES $G: \mathcal{G}^f \to \mathbb{Q}[[q]]$

2023

Part 2
- balanced setup $\dfrac{(\mathbb{Q}\langle B\rangle^0, *_b)}{\langle \tau(w)-w\rangle}$
- Racinet approach for $\mathcal{G}^f$, q-Ihara bracket

1.3 FMES

Set $A^{bi} = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \,\middle|\, k \geq 1, d \geq 0 \right\}$ and define on $\mathbb{Q}\langle A^{bi}\rangle$ the stuffle product $*$ by $1 * w = w * 1 = w$ and

$$aw * bv = a(w * bv) + b(aw * v) + (a \diamond b)(w * v),$$

where $a, b \in A^{bi}$, w, v words and $\diamond$ is the product on A^{bi} given by $\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 \\ d_1+d_2 \end{bmatrix}$

- $(\mathbb{Q}\langle A^{bi}\rangle, *)$ is a commutative $\mathbb{Q}$-algebra.
- For a word in $\mathbb{Q}\langle A^{bi}\rangle$ we write

$$\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} := \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} \cdots \begin{bmatrix} k_r \\ d_r \end{bmatrix}.$$

- Now consider the generating series

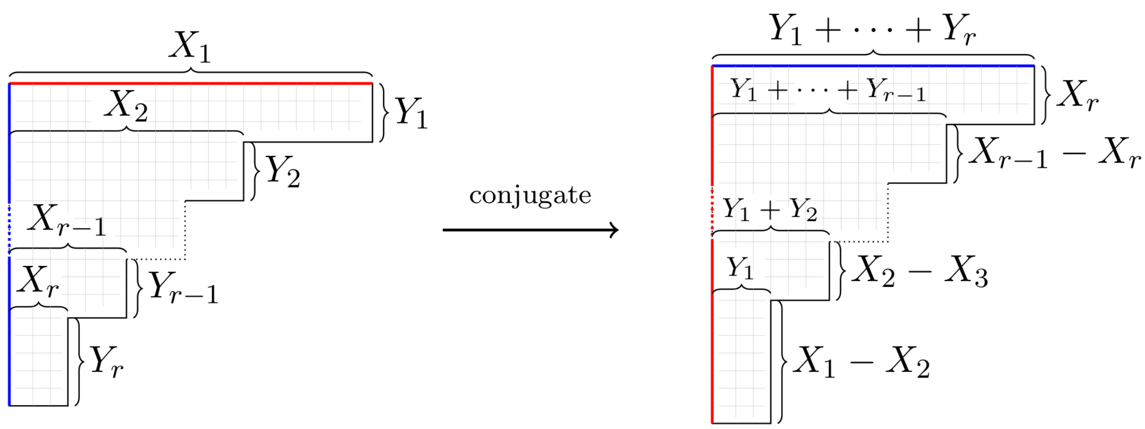
$$\mathfrak{A}\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} := \sum_{\substack{k_1, \dots, k_r \geq 1 \\ d_1, \dots, d_r \geq 0}} \begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} X_1^{k_1 - 1} \cdots X_r^{k_r - 1} \frac{Y_1^{d_1}}{d_1!} \cdots \frac{Y_r^{d_r}}{d_r!}$$

to define the swap as the $\mathbb{Q}$-linear map

$$\sigma : \mathbb{Q}\langle A\rangle \longrightarrow \mathbb{Q}\langle A\rangle$$

given by

$$\sigma\left(\mathfrak{A}\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix}\right) = \mathfrak{A}\begin{pmatrix} Y_1 + \dots + Y_r, \dots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \dots, X_1 - X_2 \end{pmatrix}$$



<u>Definition</u> The algebra of FMES is defined by

$$\mathcal{G}^f = \frac{(\mathbb{Q}\langle A^{bi}\rangle, *)}{\langle \sigma(w) - w \mid w \in \mathbb{Q}\langle A^{bi}\rangle\rangle}.$$

- We write $G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$ for the class of $\begin{bmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{bmatrix}$.
- We set $G^f(k_1,\dots,k_r) := G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$.