Formal multiple Eisenstein series and formal multiple zeta values

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$$
\begin{tikzcd}
\mathcal{E}^{\mathfrak{f}} \arrow[r] \arrow[d, two heads, "\pi"'] & \mathcal{E} \arrow[d, two heads] & \zeta(k_1,\dots,k_r) + \sum_{n>0} a_n q^n \arrow[d, mapsto] \\
\mathcal{Z}^{\mathfrak{f}} \arrow[r] & \mathcal{Z} & \zeta(k_1,\dots,k_r)
\end{tikzcd}
$$

j.w. J.W. van Ittersum & Nils Matthes, Annika Burmester, Ulf Kühn & Nils Matthes

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Formal world | **"Real world"**

Formal multiple Eisenstein series $\mathrm{G}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r}$

partial derivates of Multiple Eisenstein series ????

$\mathrm{G}(k_1,\ldots,k_r) := \mathrm{G}\binom{k_1,\ldots,k_r}{0,\ldots,0}$

$k_1,\ldots k_r \geq 1$

Stuffle regularized Multiple Eisenstein series (?)

$k_1,\ldots k_r \geq 1$

$\mathrm{G}(k_1,\ldots,k_r)$

$k_1 \geq 3, k_2,\ldots,k_r \geq 2$

Multiple Eisenstein series

$$\mathbb{G}_{k_1,\ldots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}}$$

$k_1 \geq 3, k_2,\ldots,k_r \geq 2$

abs: absolutely convergent

Formal modular forms

Modular forms

$$\begin{tikzcd}
\mathcal{G}^{\mathfrak{f}} \arrow[r, "\overset{?}{\cong}"] & \mathcal{G} \\
\mathcal{E}^{\mathfrak{f}} \arrow[u, hook, "\overset{?}{\cong}"] \arrow[r, "\overset{?}{\cong}"] & \mathcal{E} \arrow[u, hook] \\
\mathcal{E}^{\mathfrak{f},\mathrm{abs}} \arrow[u, hook] \arrow[r, "\overset{?}{\cong}"] & \mathcal{E}^{\mathrm{abs}} \arrow[u, hook] \\
\mathcal{M}^{\mathfrak{f}} \arrow[u, hook] \arrow[r, "\overset{?}{\cong}"] & \mathcal{M} \arrow[u, hook]
\end{tikzcd}$$

From formal multiple Eisenstein series to formal multiple zeta values

$$\mathrm{G}(k_1,\ldots,k_r)$$

$$\big\downarrow$$

Formal constant term map π

$$\zeta^{\mathfrak{f}}(k_1,\ldots,k_r)$$

$$\begin{tikzcd} \mathcal{E}^{\mathfrak{f}} \arrow[r, "\overset{?}{\cong}"] \arrow[d, two heads] & \mathcal{E} \arrow[d, two heads] \\ \mathcal{Z}^{\mathfrak{f}} \arrow[r, "\overset{?}{\cong}"] & \mathcal{Z} \end{tikzcd}$$

$$\zeta(k_1,\ldots,k_r)+\sum_{n>0} a_n q^n$$

$$\big\downarrow$$

Projection to the constant term

$$\zeta(k_1,\ldots,k_r)$$

$$\mathcal{Z}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}} / N$$

N : “relations satisfied by MZV which are not satisfied by MES”

① Multiple zeta values

Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs

MZVs satisfy the **double shuffle relations**, e.g. for $k_1, k_2 \geq 2$ we have

$$\begin{aligned}\zeta(k_1) \cdot \zeta(k_2) &= \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2) \\ &= \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j)\,.\end{aligned}$$

Let $\tau \in \mathbb{H}$. We define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ for two lattice points $m_1\tau + n_1, m_2\tau + n_2 \in \mathbb{Z}\tau + \mathbb{Z}$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2)\,.$$

Definition

For integers $k_1 \geq 3, k_2, \dots, k_r \geq 2$, we define the **multiple Eisenstein series** by

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}$$

and we denote the $\mathbb{Q}$-vector space spanned them by (abs = absolutely convergent)

$$\mathcal{E}^{\text{abs}} = \langle \mathbb{G}_{k_1,\dots,k_r} \mid r \geq 0, k_1 \geq 3, k_2, \dots, k_r \geq 2 \rangle_{\mathbb{Q}}\,.$$

The space $\mathcal{E}$ is an $\mathbb{Q}$-algebra, since we can express the product by the **harmonic product** formula, e.g.

$$\mathbb{G}_4(\tau) \cdot \mathbb{G}_3(\tau) = \mathbb{G}_{4,3}(\tau) + \mathbb{G}_{3,4}(\tau) + \mathbb{G}_7(\tau)\,.$$

Definition

For $k_1, \dots, k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \cdots + m_r n_r} .$$

② Multiple Eisenstein series - Fourier expansion

Definition

For $k_1, \dots, k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

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Theorem (Gangl-Kaneko-Zagier 2006 ($r = 2$), B. 2012 ($r \geq 2$))

The multiple Eisenstein series $\mathbb{G}_{k_1, \dots, k_r}(\tau)$ have a Fourier expansion of the form

$$\mathbb{G}_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n > 0} a_n q^n \qquad \left(q = e^{2\pi i \tau}\right)$$

and they can be written explicitly as a $\mathcal{Z}[2\pi i]$-linear combination of q-analogues of multiple zeta values g. In particular, $a_n \in \mathcal{Z}[2\pi i]$.

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Examples

$$\mathbb{G}_k(\tau) = \zeta(k) + (-2\pi i)^k g(k)\,,$$

$$\mathbb{G}_{3,2}(q) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3,2)\,.$$

② Multiple Eisenstein series - Extended definitions

There are different ways to extend the definition of $\mathbb{G}_{k_1,\dots,k_r}$ to $k_1,\dots,k_r \geq 1$

- Formal double zeta space realization $\mathbb{G}_{r,s}$ (Gangl-Kaneko-Zagier, 2006)

$$\mathbb{G}_{k_1}\cdot\mathbb{G}_{k_2} + (\delta_{k_1,2}+\delta_{k_2,2})\frac{\mathbb{G}'_{k_1+k_2-2}}{2(k_1+k_2-2)} = \mathbb{G}_{k_1,k_2}+\mathbb{G}_{k_2,k_1}+\mathbb{G}_{k_1+k_2}$$
$$= \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\mathbb{G}_{j,k_1+k_2-j}\,, \qquad (k_1+k_2\geq 3)\,.$$

- Shuffle regularized multiple Eisenstein series $\mathbb{G}^{\sqcup\!\sqcup}_{k_1,\dots,k_r}$ (B.-Tasaka, 2017).
- Harmonic regularized multiple Eisenstein series $\mathbb{G}^{*}_{k_1,\dots,k_r}$ (B., 2019 + ϵ). This gives a possible definition of an algebra $\mathcal{E} = \langle \mathbb{G}^{*}_{k_1,\dots,k_r} \mid r\geq 1, k_1,\dots,k_r \geq 1\rangle_{\mathbb{Q}}$ fitting into our picture.

Observation

- No version of these objects satisfy the double shuffle relations for all indices/weights.
- The derivative is always somewhere as an extra term.
- **What is the "correct" family of relations for multiple Eisenstein series (and their derivatives)?**

Multiple Eisenstein series

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}$$

$$\mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_n q^n$$

$$\mathbb{G}_4(\tau) \cdot \mathbb{G}_3(\tau) = \mathbb{G}_{4,3}(\tau) + \mathbb{G}_{3,4}(\tau) + \mathbb{G}_7(\tau)$$

$$\mathbb{G}_{3,2}(q) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3,2)$$

MZV $\zeta(k_1,\dots,k_r) \in \mathbb{R}$

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)\,,$$
$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$$

qMZV $g(k_1,\dots,k_r) \in \mathbb{Q}[[q]]$

$$g(2)g(3) = g(2,3) + g(3,2) + g(5) - \frac{1}{12} g(3)\,,$$
$$g(2)g(3) = g(2,3) + 3g(3,2) + 6g(4,1) - 3g(4) + q\frac{d}{dq} g(3)\,.$$

Define the alphabet A by

$$A = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \mid k \geq 1,\ d \geq 0 \right\}.$$

On $\mathbb{Q}A$ we define the product $\diamond$ for $k_1, k_2 \geq 1$ and $d_1, d_2 \geq 0$ by

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ d_1 + d_2 \end{bmatrix}.$$

This gives a commutative non-unital $\mathbb{Q}$-algebra $(\mathbb{Q}A, \diamond)$.

Definition

Define the **quasi-shuffle product** $*$ on $\mathbb{Q}\langle A\rangle$ as the $\mathbb{Q}$-bilinear product, which satisfies $1 * w = w * 1 = w$ for any word $w \in \mathbb{Q}\langle A\rangle$ and

$$aw * bv = a(w * bv) + b(aw * v) + (a \diamond b)(w * v)$$

for any letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A\rangle$.

Proposition

$(\mathbb{Q}\langle A\rangle, *)$ is a commutative $\mathbb{Q}$-algebra.

- For $k_1, \ldots, k_r \geq 1, d_1, \ldots, d_r \geq 0$ we use the following notation to write words in $\mathbb{Q}\langle A \rangle$:

$$\begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix} := \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \ldots \begin{bmatrix} k_r \\ d_r \end{bmatrix}.$$

- **weight:** $k_1 + \cdots + k_r + d_1 + \cdots + d_r$
- **depths:** r

In smallest depths the quasi-shuffle product is given by

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1, k_2 \\ d_1, d_2 \end{bmatrix} + \begin{bmatrix} k_2, k_1 \\ d_2, d_1 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 \\ d_1 + d_2 \end{bmatrix},$$

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2, k_3 \\ d_2, d_3 \end{bmatrix} = \begin{bmatrix} k_1, k_2, k_3 \\ d_1, d_2, d_3 \end{bmatrix} + \begin{bmatrix} k_2, k_1, k_3 \\ d_2, d_1, d_3 \end{bmatrix} + \begin{bmatrix} k_2, k_3, k_1 \\ d_2, d_3, d_1 \end{bmatrix} + \begin{bmatrix} k_1 + k_2, k_3 \\ d_1 + d_2, d_3 \end{bmatrix} + \begin{bmatrix} k_1, k_2 + k_3 \\ d_1, d_2 + d_3 \end{bmatrix}.$$

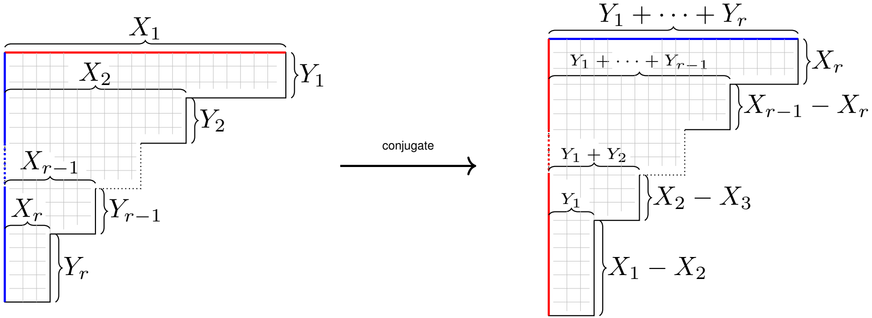
We define in depth $r \geq 1$ by the following formal power series in $\mathbb{Q}\langle A\rangle[[X_1, Y_1, \ldots, X_r, Y_r]]$

$$\mathfrak{A}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} := \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} \begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix} X_1^{k_1-1} \ldots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \ldots \frac{Y_r^{d_r}}{d_r!} \,.$$

With this the quasi-shuffle product in smallest depths reads

$$\mathfrak{A}\binom{X_1}{Y_1} * \mathfrak{A}\binom{X_2}{Y_2} = \mathfrak{A}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{A}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{A}\binom{X_1}{Y_1+Y_2} - \mathfrak{A}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2} \,.$$

The conjugation of a Young diagram with $X_1Y_1 + \cdots + X_rY_r$ boxes and r stairs:



$$\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} \longmapsto \begin{pmatrix} Y_1 + \cdots + Y_r, \ldots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \ldots, X_1 - X_2 \end{pmatrix}$$

Definition

We define the **swap** as the linear map $\sigma : \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A\rangle$ by $\sigma(1) = 1$ and for $r \geq 1$ on the generators of $\mathbb{Q}\langle A\rangle$ by

$$\sigma\left(\mathfrak{A}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}\right) := \mathfrak{A}\binom{Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2},$$

where σ is applied coefficient-wise on the left, i.e. $\sigma\left(\begin{bmatrix}k_1,\dots,k_r\\d_1,\dots,d_r\end{bmatrix}\right)$ is defined as the coefficient of $X_1^{k_1-1}\dots X_r^{k_r-1}\frac{Y_1^{d_1}}{d_1!}\dots\frac{Y_r^{d_r}}{d_r!}$ on the right-hand side.

$$\sigma\left(\begin{bmatrix}k\\d\end{bmatrix}\right) = \frac{d!}{(k-1)!}\begin{bmatrix}d+1\\k-1\end{bmatrix}, \qquad (k \geq 1, d \geq 0).$$

Define S as the ideal in $(\mathbb{Q}\langle A\rangle, *)$ generated by all $\sigma(w) - w$ for $w \in \mathbb{Q}\langle A\rangle$, i.e.

$$S = \langle \sigma(w) - w \mid w \in \mathbb{Q}\langle A\rangle\rangle_{\mathbb{Q}} * \mathbb{Q}\langle A\rangle \,.$$

Definition

The algebra of **formal multiple Eisenstein series** is defined by

$$\mathcal{G}^{\mathfrak{f}} = {\mathbb{Q}\langle A\rangle}\big/{S}$$

and we denote the class of a word $\begin{bmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{bmatrix}$ by $\mathrm{G}\begin{pmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{pmatrix}$.

We obtain a commutative $\mathbb{Q}$-algebra $(\mathcal{G}^{\mathfrak{f}}, *)$, where each element is swap invariant. We write

$$\mathfrak{G}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} := \sum_{\substack{k_1,\dots,k_r\geq 1\\ d_1,\dots,d_r\geq 0}} \mathrm{G}\binom{k_1,\dots,k_r}{d_1,\dots,d_r} X_1^{k_1-1}\dots X_r^{k_r-1}\frac{Y_1^{d_1}}{d_1!}\dots\frac{Y_r^{d_r}}{d_r!}\,.$$

Since the formal multiple Eisenstein series are swap invariant and their product is given by $*$ we have in particular

$$\mathfrak{G}\binom{X_1}{Y_1} = \mathfrak{G}\binom{Y_1}{X_1}\,,$$

$$\mathfrak{G}\binom{X_1,X_2}{Y_1,Y_2} = \mathfrak{G}\binom{Y_1+Y_1,Y_1}{X_2,X_1-X_2}\,,$$

$$\mathfrak{G}\binom{X_1}{Y_1} * \mathfrak{G}\binom{X_2}{Y_2} = \mathfrak{G}\binom{X_1,X_2}{Y_1,Y_2} + \mathfrak{G}\binom{X_2,X_1}{Y_2,Y_1} + \frac{\mathfrak{G}\binom{X_1}{Y_1+Y_2} - \mathfrak{G}\binom{X_2}{Y_1+Y_2}}{X_1-X_2}\,.$$

Let $\partial : (\mathbb{Q}A, \diamond) \to (\mathbb{Q}A, \diamond)$ be the derivation defined for $k \geq 1, d \geq 0$ by

$$\partial\left(\begin{bmatrix} k \\ d \end{bmatrix}\right) = k \begin{bmatrix} k+1 \\ d+1 \end{bmatrix}.$$

This gives a derivation on $\mathbb{Q}\langle A \rangle$ (with respect to the concatenation product), defined by

$$\partial\left(\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix}\right) = \sum_{j=1}^{r} k_j \begin{bmatrix} k_1, \dots, k_j + 1, \dots, k_r \\ d_1, \dots, d_j + 1, \dots, d_r \end{bmatrix}.$$

Lemma

- ∂ is a derivation on $(\mathbb{Q}\langle A\rangle, *)$.
- The derivation ∂ commutes with the swap, i.e. $\partial\sigma = \sigma\partial$.

Theorem

∂ is a derivation on $(\mathcal{G}^{\mathfrak{f}}, *)$.

$$\partial\left(\mathrm{G}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}\right) = \sum_{j=1}^{r} k_j\,\mathrm{G}\binom{k_1,\dots,k_j+1,\dots,k_r}{d_1,\dots,d_j+1,\dots,d_r}.$$

Conjecture

There exist a unique derivation $\mathfrak{d}$ on $(\mathbb{Q}\langle A\rangle, *)$ such that

- $\mathfrak{d}$ commutes with σ.
- The triple $(\partial, W, \mathfrak{d})$ satisfies the commutation relations of an $\mathfrak{sl}_2$-triple, i.e.

$$[W,\partial] = 2\partial, \quad [W,\mathfrak{d}] = -2\mathfrak{d}, \qquad [\mathfrak{d},\partial] = W\,,$$

where W is the weight operator, multiplying a word $\left[\begin{smallmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{smallmatrix}\right]$ by its weight $k_1 + \ldots + k_r + d_1 + \ldots + d_r$.

This would imply an $\mathfrak{sl}_2$-action on $\mathcal{G}^{\mathfrak{f}}$. In depth one this derivation seems to be given by

$$\mathfrak{d}\,\mathrm{G}\binom{k}{d} = d\cdot \mathrm{G}\binom{k-1}{d-1} - \frac{1}{2}\delta_{k+d,2}\,,$$

which correspond to the classical derivation for quasi-modular forms (the derivative with respect to G_2).
For any $r \geq 1$ we have explicit (conjectured) formulas for $\mathfrak{d}\,\mathrm{G}\left(\begin{smallmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{smallmatrix}\right)$. (See bonus slide for the $r=2$ case)

On $\mathbb{Q}\langle A\rangle$ we can define another product $\sqcup\!\sqcup$ by $w \sqcup\!\sqcup v = \sigma(\sigma(w) * \sigma(v))$ for $w, v \in \mathbb{Q}\langle A\rangle$. For any $f, g \in \mathcal{G}^{\mathfrak{f}}$ we have $f \sqcup\!\sqcup g - f * g = 0$.

Proposition

For $k_1, k_2 \geq 1, d_1, d_2 \geq 0$ we have

$$\begin{aligned}
\mathrm{G}\binom{k_1}{d_1}\mathrm{G}\binom{k_2}{d_2} &= \mathrm{G}\binom{k_1, k_2}{d_1, d_2} + \mathrm{G}\binom{k_2, k_1}{d_2, d_1} + \mathrm{G}\binom{k_1 + k_2}{d_1 + d_2} \\
&= \sum_{\substack{l_1+l_2=k_1+k_2 \\ e_1+e_2=d_1+d_2}} \left(\binom{l_1-1}{k_1-1}\binom{d_1}{e_1}(-1)^{d_1-e_1} + \binom{l_1-1}{k_2-1}\binom{d_2}{e_1}(-1)^{d_2-e_1}\right)\mathrm{G}\binom{l_1, l_2}{e_1, e_2} \\
&\quad + \frac{d_1! d_2!}{(d_1+d_2+1)!}\binom{k_1+k_2-2}{k_1-1}\mathrm{G}\binom{k_1+k_2-1}{d_1+d_2+1},
\end{aligned}$$

where we sum over all $l_1, l_2 \geq 1$ and $e_1, e_2 \geq 0$ in the second expression

The special case $d_1 = d_2 = 0$ is similar to the double shuffle relations of MZV.

Most of the relations we will obtain are among G, where the bottom entries are zero. For shorter notation we will denote these for $k_1,\dots,k_r \geq 1$ by

$$\mathrm{G}(k_1,\dots,k_r) := \mathrm{G}\binom{k_1,\dots,k_r}{0,\dots,0}.$$

Instead of $*$ we will just write products of G (i.e. this will not denote the concatenation of words)

Example

$$\begin{aligned}\mathrm{G}(2)\,\mathrm{G}(3) =&\ \mathrm{G}(2,3)+\mathrm{G}(3,2)+\mathrm{G}(5)\\ =&\ \mathrm{G}(2,3)+3\,\mathrm{G}(3,2)+6\,\mathrm{G}(4,1)+3\,\mathrm{G}\binom{4}{1}.\end{aligned}$$

Compare this to the previous example of multiple zeta values. Also notice: $3\,\mathrm{G}\binom{4}{1} = \partial\,\mathrm{G}(3)$.

Theorem (B.-van Ittersum-Matthes 2021+)

For all $k_1, k_2 \geq 1$ with $k = k_1 + k_2 \geq 4$ even we have

$$\frac{1}{2}\left(\binom{k_1+k_2}{k_2} - (-1)^{k_1}\right) \mathrm{G}(k) = \sum_{\substack{j=2 \\ j \text{ even}}}^{k-2} \left(\binom{k-j-1}{k_1-1} + \binom{k-j-1}{k_2-1} - \delta_{j,k_1}\right) \mathrm{G}(j)\,\mathrm{G}(k-j)$$
$$+ \frac{1}{2}\left(\binom{k-3}{k_1-1} + \binom{k-3}{k_2-1} + \delta_{k_1,1} + \delta_{k_2,1}\right) \mathrm{G}\binom{k-1}{1}.$$

Proof sketch:

- Define an action of the group ring $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$ on the generating series in depth two.
- Above equality follows by describing the double shuffle relations in terms of this action together with some identities in $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$.

(See bonus slides for details)

Corollary

- For even $k \geq 4$ we have

$$\frac{k+1}{2}\,\mathrm{G}(k) = \mathrm{G}\binom{k-1}{1} + \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 2 \text{ even}}} \mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

- For all even $k \geq 6$ we have

$$\frac{(k+1)(k-1)(k-6)}{12}\,\mathrm{G}(k) = \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 4 \text{ even}}} (k_1-1)(k_2-1)\;\mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

Example

$$\mathrm{G}(8) = \frac{6}{7}\,\mathrm{G}(4)^2, \quad \mathrm{G}(10) = \frac{10}{11}\,\mathrm{G}(4)\,\mathrm{G}(6), \quad \mathrm{G}(12) = \frac{84}{143}\,\mathrm{G}(4)\,\mathrm{G}(8) + \frac{50}{143}\,\mathrm{G}(6)^2\,.$$

Notice that for $k \geq 3$ we have $\frac{1}{k-2}\, \mathrm{G}\binom{k-1}{1} = \partial\, \mathrm{G}(k-2) = \mathrm{G}'(k-2)$.

Corollary

- For $m \geq 1$ we have $\mathrm{G}(2m) \in \mathbb{Q}[\mathrm{G}(2), \mathrm{G}'(2), \mathrm{G}''(2)] = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}(4), \mathrm{G}(6)]$ and

$$\mathrm{G}(2m) = -\frac{B_{2m}}{2(2m)!}(-24\, \mathrm{G}(2))^m + \text{terms with } \mathrm{G}'(2) \text{ and } \mathrm{G}''(2)\,.$$

- For $m \geq 2$ we have $\mathrm{G}(2m) \in \mathbb{Q}[\mathrm{G}(4), \mathrm{G}(6)]$.

Compare the first part with the formula by Euler for Riemann zeta values: $\zeta(2m) = -\frac{B_{2m}}{2(2m)!}(-24\zeta(2))^m$.

Example

$$\mathrm{G}(4) = \frac{2}{5}\, \mathrm{G}(2)^2 + \frac{1}{5}\, \mathrm{G}'(2)\,,$$
$$\mathrm{G}(6) = \frac{8}{35}\, \mathrm{G}(2)^3 + \frac{6}{35}\, \mathrm{G}(2)\, \mathrm{G}'(2) + \frac{1}{70}\, \mathrm{G}''(2)\,.$$

$$\mathcal{E}^{\mathfrak{f}} = \mathbb{Q} + \langle \mathrm{G}(k_1, \dots, k_r) \mid r \geq 1, k_1, \dots, k_r \geq 1 \rangle_{\mathbb{Q}} \subset \mathcal{G}^{\mathfrak{f}}\,.$$

By the definition of the quasi-shuffle product, it is easy to see that $\left(\mathcal{E}^{\mathfrak{f}}, *\right)$ is a subalgebra of $\left(\mathcal{G}^{\mathfrak{f}}, *\right)$.
Applying ∂ to the generators of $\mathcal{E}^{\mathfrak{f}}$ gives

$$\partial\left(\mathrm{G}(k_1, \dots, k_r)\right) = \sum_{j=1}^{r} k_j \, \mathrm{G}\binom{k_1, \dots, k_j + 1, \dots, k_r}{0, \dots, 1, \dots, 0}\,.$$

Proposition (B.-van Ittersum 2021+)

$\mathcal{E}^{\mathfrak{f}}$ is closed under ∂.

$$\mathcal{E}^{\mathfrak{f}} = \mathbb{Q} + \langle \mathrm{G}(k_1, \dots, k_r) \mid r \geq 1, k_1, \dots, k_r \geq 1 \rangle_{\mathbb{Q}} \subset \mathcal{G}^{\mathfrak{f}} .$$

By the definition of the quasi-shuffle product, it is easy to see that $(\mathcal{E}^{\mathfrak{f}}, *)$ is a subalgebra of $(\mathcal{G}^{\mathfrak{f}}, *)$.
Applying ∂ to the generators of $\mathcal{E}^{\mathfrak{f}}$ gives

$$\partial\left(\mathrm{G}(k_1, \dots, k_r)\right) = \sum_{j=1}^{r} k_j \, \mathrm{G}\begin{pmatrix} k_1, \dots, k_j + 1, \dots, k_r \\ 0, \dots, 1, \dots, 0 \end{pmatrix} .$$

Proposition (B.-van Ittersum 2021+)

$\mathcal{E}^{\mathfrak{f}}$ is closed under ∂.

Conjecture (B. 2015)

We have $\mathcal{E}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}}$.

Question

What are the "constant terms" of formal multiple Eisenstein series?

- To define formal cusp forms, we want to determine the projection onto the constant term of formal multiple Eisenstein series.
- This leads to the question of which relations are additionally satisfied for MZV compared to MES.
- This will give a definition of formal multiple zeta values.
- The following construction is motivated/inspired by a conjectural construction of combinatorial multiple Eisenstein series together with their behavior as $q \to 1$.

We define the following two subsets of the alphabet A

$$A_0 = \left\{ \begin{bmatrix} k \\ 0 \end{bmatrix} \mid k \geq 1 \right\}, \qquad A^1 = \left\{ \begin{bmatrix} 1 \\ d \end{bmatrix} \mid d \geq 0 \right\}.$$

With this we define the following ideal in $(\mathbb{Q}\langle A\rangle, *)$ generated by the set $A^*\backslash(A^1)^*(A_0)^*$

$$N = \big(A^*\backslash(A^1)^*(A_0)^* \big)_{\mathbb{Q}\langle A\rangle} ,$$

The elements in $A^*\backslash(A^1)^*(A_0)^*$ are exactly those elements which are <u>not</u> of the form

$$\begin{bmatrix} 1, \dots, 1, k_1, \dots, k_r \\ d_1, \dots, d_s, 0, \dots, 0 \end{bmatrix}.$$

Definition

The algebra of **formal multiple zeta values** is defined by

$$\mathcal{Z}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}} \big/ N \,.$$

We denote the canonical projection by

$$\pi : \mathcal{G}^{\mathfrak{f}} \longrightarrow \mathcal{Z}^{\mathfrak{f}} \,.$$

This map can be seen as the formal version of the "projection onto the constant term". We refer to π as the **formal constant term map**.

Claim: The ideal N captures the additional relations satisfied by multiple zeta values, which are not satisfied by multiple Eisenstein series.

Proposition (B.-van Ittersum-Matthes 2021+)

The map $\pi_{|\mathcal{E}^{\mathfrak{f}}} : \mathcal{E}^{\mathfrak{f}} \to \mathcal{Z}^{\mathfrak{f}}$ is surjective.

Definition

For $k_1, \dots, k_r \geq 1$ we define the **formal multiple zeta value** $\zeta^{\mathfrak{f}}(k_1, \dots, k_r)$ by

$$\zeta^{\mathfrak{f}}(k_1, \dots, k_r) = \pi(\mathrm{G}(k_1, \dots, k_r)) \,.$$

Proposition

We have $\partial \mathcal{G}^{\mathfrak{f}} \subset \ker(\pi)$.

Applying the formal constant term map to the previously proven results yields the following:

Corollary

- *(Double shuffle relations in depth two)* For $k_1, k_2 \geq 1$ we have

$$\begin{aligned}\zeta^{\mathfrak{f}}(k_1)\zeta^{\mathfrak{f}}(k_2) &= \zeta^{\mathfrak{f}}(k_1,k_2) + \zeta^{\mathfrak{f}}(k_2,k_1) + \zeta^{\mathfrak{f}}(k_1+k_2)\\ &= \sum_{l_1+l_2=k_1+k_2} \left(\binom{l_1-1}{k_1-1} + \binom{l_1-1}{k_2-1} \right) \zeta^{\mathfrak{f}}(l_1,l_2) + \delta_{k_1+k_2,2}\zeta^{\mathfrak{f}}(2)\,.\end{aligned}$$

  In particular we obtain the relation $\zeta^{\mathfrak{f}}(3) = \zeta^{\mathfrak{f}}(2,1)$ by taking $k_1 = 1, k_2 = 2$.

- *(Euler relation)* For $m \geq 1$ we have

$$\zeta^{\mathfrak{f}}(2m) = -\frac{B_{2m}}{2(2m)!}\left(-24\zeta^{\mathfrak{f}}(2)\right)^m\,.$$

Theorem (B.-van Ittersum-Matthes 2021+)

The formal multiple zeta values satisfy exactly the extended double shuffle relations.

- Our formal multiple zeta values are isomorphic (after switching to the shuffle regularization) to the classical definition of formal multiple zeta values (Racinet).
- There is a 1:1 correspondence between objects satisfying the extended double shuffle relations and the objects satisfying the relations in $\mathcal{Z}^{\mathfrak{f}}$.

In contrast to the analytic case, we start by defining formal quasi-modular forms before formal modular forms.

Definition

We define the algebra of **formal quasi-modular forms** $\widetilde{\mathcal{M}}^{\mathfrak{f}}$ as the smallest subalgebra of $\mathcal{G}^{\mathfrak{f}}$ which satisfies the following two conditions

- $\mathrm{G}(2) \in \widetilde{\mathcal{M}}^{\mathfrak{f}}$.
- $\widetilde{\mathcal{M}}^{\mathfrak{f}}$ is closed under ∂.

Proposition

- We have $\widetilde{\mathcal{M}}^{\mathfrak{f}} = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}(4), \mathrm{G}(6)] = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}'(2), \mathrm{G}''(2)]$.
- The Ramanujan differential equations are satisfied:

$$\mathrm{G}'(2) = 5\,\mathrm{G}(4) - 2\,\mathrm{G}(2)^2\,,$$
$$\mathrm{G}'(4) = 8\,\mathrm{G}(6) - 14\,\mathrm{G}(2)\,\mathrm{G}(4)\,,$$
$$\mathrm{G}'(6) = \frac{120}{7}\,\mathrm{G}(4)^2 - 12\,\mathrm{G}(2)\,\mathrm{G}(6)\,.$$

- The Chazy equation is satisfied

$$\mathrm{G}'''(2) + 24\,\mathrm{G}(2)\,\mathrm{G}''(2) - 36\,\mathrm{G}'(2)^2 = 0\,.$$

$$\frac{k+1}{2}\,\mathrm{G}(k) = \mathrm{G}\binom{k-1}{1} + \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 2 \text{ even}}} \mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

Definition

- The algebra of **formal modular forms** $\mathcal{M}^{\mathfrak{f}}$ is defined as the subalgebra of $\mathcal{G}^{\mathfrak{f}}$ generated by all $\mathrm{G}(k)$ with $k \geq 4$ even. (Alternative definition: $\mathcal{M}^{\mathfrak{f}} = \ker \mathfrak{d}_{|\widetilde{\mathcal{M}}^{\mathfrak{f}}}$)
- We define the algebra of **formal cusp forms** by $\mathcal{S}^{\mathfrak{f}} = \ker \pi_{|\mathcal{M}^{\mathfrak{f}}}$.

The first example of a non-zero formal cusp form appears in weight 12 and we write

$$\Delta^{\mathfrak{f}} = 2400 \cdot 6! \cdot \mathrm{G}(4)^3 - 420 \cdot 7! \cdot \mathrm{G}(6)^2 \,.$$

Proposition

- We have $\mathcal{M}^{\mathfrak{f}} = \mathbb{Q}[\mathrm{G}(4), \mathrm{G}(6)]$ and $\mathcal{M}^{\mathfrak{f}}_k = \mathbb{Q}\,\mathrm{G}(k) \oplus \mathcal{S}^{\mathfrak{f}}_k$.
- We have $\Delta^{\mathfrak{f}} \in \mathcal{S}^{\mathfrak{f}}_{12}$ and $\partial \Delta^{\mathfrak{f}} = -24\,\mathrm{G}(2)\Delta^{\mathfrak{f}}$.

$$\frac{1}{432}\Delta^{\mathfrak{f}} = 48\,\mathrm{G}(2)^2\,\mathrm{G}'(2)^2 + 32\,\mathrm{G}'(2)^3 - 32\,\mathrm{G}(2)^3\,\mathrm{G}''(2) - 24\,\mathrm{G}(2)\,\mathrm{G}'(2)\,\mathrm{G}''(2) - \mathrm{G}''(2)^2 \,.$$

Work in progress/Outlook

There are more aspects of formal multiple Eisenstein series;

- Connection to the formal double zeta space of Gangl, Kaneko & Zagier. (see bonus slides)
- Rankin-Cohen brackets as a consequence of the $\mathfrak{sl}_2$-action on $\widetilde{\mathcal{M}}^{\mathfrak{f}}$.
- A formal version of "vanishing order at $i\infty$" by considering the kernels of

$$\pi_a : \mathcal{G}^{\mathfrak{f}} \longrightarrow \mathcal{G}^{\mathfrak{f}} \big/ N^a , \qquad (a \geq 1)$$

- Miller basis, Dimension formulas.

Not clear: How to formalize other important structures, such as Hecke operators ?

Happy 10^{th} birthday Kansai MZV Meeting!

Formal world

"Real world"

$$
\begin{tikzcd}
\mathcal{G}^{\mathfrak{f}} \arrow[r, "\overset{?}{\cong}"] & \mathcal{G} \\
\mathcal{E}^{\mathfrak{f}} \arrow[r, "\overset{?}{\cong}"] \arrow[u, hook, "\overset{?}{\cong}"] & \mathcal{E} \arrow[u, hook] \\
\mathcal{E}^{\mathfrak{f},\mathrm{abs}} \arrow[r, "\overset{?}{\cong}"] \arrow[u, hook] & \mathcal{E}^{\mathrm{abs}} \arrow[u, hook] \\
\mathcal{M}^{\mathfrak{f}} \arrow[r, "\overset{?}{\cong}"] \arrow[u, hook] & \mathcal{M} \arrow[u, hook]
\end{tikzcd}
$$

Formal multiple Eisenstein series

$$\mathrm{G}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$$

"bi" Multiple Eisenstein series

????

$$\mathrm{G}(k_1,\dots,k_r) := \mathrm{G}\binom{k_1,\dots,k_r}{0,\dots,0}$$

$$k_1,\dots k_r \geq 1$$

Stuffle regularized Multiple Eisenstein series (?)

$$k_1,\dots k_r \geq 1$$

$$\mathrm{G}(k_1,\dots,k_r)$$

$$k_1 \geq 3, k_2,\dots,k_r \geq 2$$

Multiple Eisenstein series

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}$$

$$k_1 \geq 3, k_2,\dots,k_r \geq 2$$

Formal modular forms

Modular forms

⑥ Realizations - Definition

Definition

Let A be a (differential) $\mathbb{Q}$-algebra. A **realization of $\mathcal{G}^{\mathfrak{f}}$ in A** is an (differential) algebra homomorphism

$$\varphi : \mathcal{G}^{\mathfrak{f}} \longrightarrow A\,.$$

- $A = \mathbb{R}$: Multiple zeta values (derivation = zero map).
- $A = \mathbb{Q}$: Rational solution to extended double shuffle.
- $A = \mathbb{Q}[[q]]$: Combinatorial multiple Eisenstein series (derivation = $q\frac{d}{dq}$).
- $A = \mathcal{O}(\mathbb{H})$: ("Analytical") multiple Eisenstein series (derivation = $(2\pi i)\frac{d}{d\tau}$).

Theorem (B.-van Ittersum-Matthes 2021+)

For any field A of characteristic zero, there exist a realization of $\mathcal{G}^{\mathfrak{f}}$ in A, which factors through π.

- This follows from the fact that we know that for any field A of characteristic zero, there exists a solution to the extended double shuffle relations.
- For $A = \mathbb{R}$ these are given, for example, by (harmonic regularized) multiple zeta values.
- For $A = \mathbb{Q}$, there is no explicit construction known so far for depth ≥ 4.

⑥ Realizations - Multiple zeta values II

Definition

For $k_1, \dots k_r \geq 1, d_1, \dots, d_r \geq 0$ define the q-series

$$g\binom{k_1, \dots, k_r}{d_1, \dots, d_r} = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{m_1^{d_1} n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{m_r^{d_r} n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \cdots + m_r n_r} \,.$$

Theorem (B.-van Ittersum 2021+)

The following gives a realization of $\mathcal{G}^{\mathfrak{f}}$ in $\mathbb{R}$

$$\varphi : \mathrm{G}\binom{k_1, \dots, k_r}{d_1, \dots, d_r} \longmapsto \lim_{q \to 1}^{*} (1-q)^{k_1 + \cdots + k_r + d_1 + \cdots + d_r} g\binom{k_1, \dots, k_r}{d_1, \dots, d_r} \,,$$

where $\lim^*_{q \to 1}$ is a "(harmonic) regularized limit". This realization factors through π and we have

$$\varphi(\mathrm{G}(k_1, \dots, k_r)) = \zeta^*(k_1, \dots, k_r) \,.$$

⑥ Realizations - Combinatorial MES

$$\mathfrak{G}\binom{X_1}{Y_1} = \mathfrak{G}\binom{Y_1}{X_1}, \qquad \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} = \mathfrak{G}\binom{Y_1 + Y_1, Y_1}{X_2, X_1 - X_2},$$

$$\mathfrak{G}\binom{X_1}{Y_1}\mathfrak{G}\binom{X_2}{Y_2} = \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{G}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{G}\binom{X_1}{Y_1+Y_2} - \mathfrak{G}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}.$$

Theorem (B.-Kühn-Matthes 2021+, B.-Burmester 2021+)

There exist power series $\mathfrak{G}\binom{Y_1}{X_1}, \mathfrak{G}\binom{X_1,X_2}{Y_1,Y_2} \in \mathbb{Q}[[q]][[X_1, X_2, Y_1, Y_2]]$ which satisfy the above equations and where the coefficients of $\mathfrak{G}\binom{Y_1}{X_1}$ are given by (derivatives of) Eisenstein series. (See bonus slides)

- This gives combinatorial proofs of the classical identities for quasi-modular forms.
- There exists a construction for depth ≥ 3, which conjecturally gives a realization of $\mathcal{G}^{\mathfrak{f}}$. See the talkslides of Annika Burmesters talk "Combinatorial multiple Eisenstein series" at the JENTE Seminar (https://sites.google.com/view/jente-seminar/home).

$r = 1$

$$\mathfrak{d}\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} = d_1 \begin{bmatrix} k_1 - 1 \\ d_1 - 1 \end{bmatrix} - \frac{1}{2}\delta_{k_1+d_1,2}$$

$r = 2$

$$\begin{aligned}
\mathfrak{d}\begin{bmatrix} k_1, k_2 \\ d_1, d_2 \end{bmatrix} &= d_1 \begin{bmatrix} k_1 - 1, k_2 \\ d_1 - 1, d_2 \end{bmatrix} + \frac{1}{2}\delta_{k_1,1}\left(\begin{bmatrix} k_2 - 1 \\ d_1 + d_2 \end{bmatrix} - d_1 \begin{bmatrix} k_2 \\ d_1 + d_2 - 1 \end{bmatrix}\right) - \frac{1}{2}\delta_{k_1,2}\begin{bmatrix} k_2 \\ d_1 + d_2 \end{bmatrix} \\
&+ d_2 \begin{bmatrix} k_1, k_2 - 1 \\ d_1, d_2 - 1 \end{bmatrix} - \frac{1}{2}\delta_{k_2,1}\left(\begin{bmatrix} k_1 - 1 \\ d_1 + d_2 \end{bmatrix} + d_2 \begin{bmatrix} k_1 \\ d_1 + d_2 - 1 \end{bmatrix}\right) + \frac{1}{2}\delta_{k_2,2}\begin{bmatrix} k_1 \\ d_1 + d_2 \end{bmatrix} \\
&- \frac{1}{2}\delta_{k_2+d_2,2}\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} + \frac{1}{4}\delta_{k_1+k_2+d_1+d_2,2} \,.
\end{aligned}$$

In 2006 Gangl, Kaneko and Zagier introduced for $k \geq 1$ the **formal double zeta space** in weight k as

$$\mathcal{D}_k = \left\langle Z_k, Z_{k_1,k_2}, P_{k_1,k_2} \mid k_1 + k_2 = k, k_1, k_2 \geq 1 \right\rangle_{\mathbb{Q}} \Big/ \text{(1)}$$

where they divide out the following relations for $k_1, k_2 \geq 1$

$$\begin{aligned} P_{k_1,k_2} &= Z_{k_1,k_2} + Z_{k_2,k_1} + Z_{k_1+k_2} \\ &= \sum_{l_1+l_2=k_1+k_2} \left(\binom{l_1 - 1}{k_1 - 1} + \binom{l_1 - 1}{k_2 - 1} \right) Z_{l_1,l_2} \,. \end{aligned} \qquad (1)$$

Proposition

For all $k \geq 1$ the following gives a $\mathbb{Q}$-linear map $\mathcal{D}_k \to \mathcal{G}^{\mathfrak{f}}$

$$
\begin{aligned}
Z_k &\longmapsto \mathrm{G}(k) - \delta_{k,2}\,\mathrm{G}(2)\,,\\
Z_{k_1,k_2} &\longmapsto \mathrm{G}(k_1,k_2) + \frac{1}{2}\left(\delta_{k_2,1}\,\mathrm{G}\binom{k_1}{1} - \delta_{k_1,1}\,\mathrm{G}\binom{k_2}{1} + \delta_{k_1,2}\,\mathrm{G}\binom{k_2+1}{1}\right)\,,\\
P_{k_1,k_2} &\longmapsto \mathrm{G}(k_1)\,\mathrm{G}(k_2) + \frac{1}{2}\left(\delta_{k_1,2}\,\mathrm{G}\binom{k_2+1}{1} + \delta_{k_2,2}\,\mathrm{G}\binom{k_1+1}{1}\right)\,.
\end{aligned}
$$

The double shuffle relations for formal multiple Eisenstein series in lowest depth are

$$
\begin{aligned}
P\binom{X_1, X_2}{Y_1, Y_2} &= \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{G}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{G}\binom{X_1}{Y_1+Y_2} - \mathfrak{G}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2} \\
&= \mathfrak{G}\binom{X_1+X_2, X_2}{Y_1, Y_2 - Y_1} + \mathfrak{G}\binom{X_1+X_2, X_1}{Y_2, Y_1 - Y_2} + \frac{\mathfrak{G}\binom{X_1+X_2}{Y_1} - \mathfrak{G}\binom{X_1+X_2}{Y_2}}{Y_1 - Y_2}
\end{aligned}
\tag{2}
$$

with $P\binom{X_1,X_2}{Y_1,Y_2} = \mathfrak{G}\binom{X_1}{Y_1}\mathfrak{G}\binom{X_2}{Y_2}$. Define the action of the group ring $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$ on the formal Laurent series $\mathcal{L} = \mathbb{Q}\langle A\rangle((X_1, X_2, Y_1, Y_2))$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{Z})$ and $R \in \mathcal{L}$ by

$$
R_{|\gamma}\binom{X_1, X_2}{Y_1, Y_2} = R\binom{aX_1 + bX_2, cX_1 + dX_2}{\det(\gamma)(dY_1 - cY_2), \det(\gamma)(-bY_1 + aY_2)}
$$

and then extend it linearly to all elements in $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$.

⑥ Bonus - Action of $\mathrm{Gl}_2(\mathbb{Z})$ - 2

Now define the following elements in $\mathrm{Gl}_2(\mathbb{Z})$

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The equation (2) then becomes $P = \mathfrak{G} \mid (1+\epsilon) + R^* = \mathfrak{G} \mid T(1+\epsilon) + R^{\sqcup\!\sqcup}$ with

$$R^*\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = \frac{\mathfrak{G}\binom{X_1}{Y_1+Y_2} - \mathfrak{G}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}, \quad R^{\sqcup\!\sqcup}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = \frac{\mathfrak{G}\binom{X_1+X_2}{Y_1} - \mathfrak{G}\binom{X_1+X_2}{Y_2}}{Y_1 - Y_2}.$$

Lemma

For $A = \epsilon U \epsilon$ we have

$$\mathfrak{G} \mid (1-\sigma) = P \mid (1-\delta)(1 + A - SA^2) - (R^* - R^{\sqcup\!\sqcup} \mid (T^{-1}\epsilon)) \mid (1 + A + A^2).$$

Considering the coefficients in above Lemma gives the Theorem on products of G.

⑥ Bonus - Combinatorial MES explicit

Theorem (B.-Kühn-Matthes 2021+, B.-Burmester 2021+)

The following series are swap invariant and their coefficients satisfy the quasi-shuffle product

$$\mathfrak{G}\binom{X_1}{Y_1} = \beta\binom{X_1}{Y_1} + \mathfrak{g}\binom{X_1}{Y_1},$$

$$\begin{aligned}\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} &= \beta\binom{X_1, X_2}{Y_1, Y_2} - \beta\binom{X_1 - X_2}{Y_2}\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \frac{1}{2}\mathfrak{g}\binom{X_1}{Y_1 + Y_2} \\ &\quad + \beta\binom{X_2}{Y_2}\mathfrak{g}\binom{X_1}{Y_1} + \beta\binom{X_1 - X_2}{Y_1}\mathfrak{g}\binom{X_2}{Y_1 + Y_2} + \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2}.\end{aligned}$$

Here β is a rational realization of $\mathcal{Z}^{\mathfrak{f}}$, such that the depth one objects are exactly the constant terms of the Eisenstein series G_k and

$$\mathfrak{g}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \dots, n_r > 0}} \prod_{j=1}^{r} e^{X_j n_j + Y_j m_j} q^{m_j n_j}.$$