Hamburgers, Numbers and infinite Series

JSPS Science Dialogue

$\frac{1}{2}$ π $\sqrt{2}$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Henrik Bachmann

Nagoya University

Plan of this Talk

- Hamburgers, Germany and Me
- Numbers
  - Real numbers, Algebraic numbers, Rational numbers and Transcendental numbers
- Infinite Series
  - Finite & Infinite Series
  - Geometric series and Riemann zeta values
  - Multiple zeta values
- Mathematics & Research
  - What I like about mathematics

What is this?
これは何ですか？

... and this?

Das ist ein Hamburger

This is a hamburger

Das ist ein Hamburger

This is someone from Hamburg

About me

- In 1985 I was born in **Hamburg, Germany.**
- I have an older **sister** who studied Japanese.
  姉妹
- From 2006 to 2016 I did my Bachelor, Master and Doctor in **Mathematics** at Hamburg University.
- Since April 2016 I am a JSPS Post-doctoral Fellow in **Nagoya**.
- I am interested in **Number theory**.
  数論

Where is Germany?

Where is Hamburg?

Hamburg

Old name: Hammaburg

Hamma = Something at a river

Burg = Castle

Moin Moin!

Special phrase from Hamburg for こんにちわ

What does this have to do with  ?

- A lot of people moved from Germany to the USA.

- There the people from Hamburg sold a north German food: A piece of meat inside a small bread.

- Later the American people called these breads “Hamburger”.

What do you know about Germany?







Germany & Japan

82.175.684

357.375 km²

1. Berlin (3.520.031)
2. **Hamburg (1.787.408)**
3. München (1.450.381)
4. Köln (1.060.582)

...

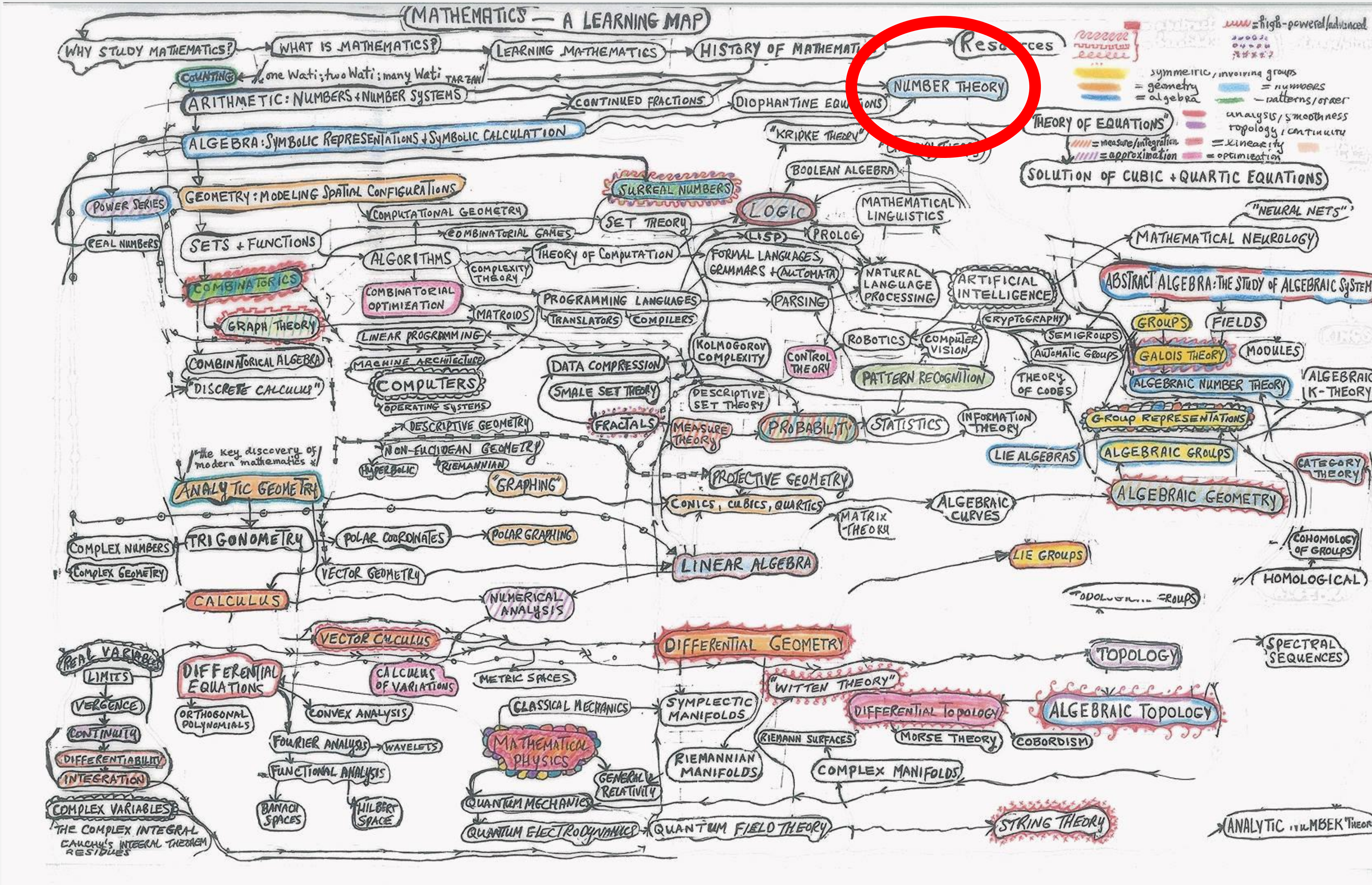
127.110.047

377.835 km²

1. Tokyo (9.375.104)
2. Yokohama (3.732.616)
3. Osaka (2.705.262)
4. **Nagoya (2.302.696)**

...

Mathematics overview



What is number theory (数論) ?

- Number theory is a branch of pure mathematics devoted primarily to the study of the integers (numbers).

- It is sometimes called "The Queen of Mathematics“.

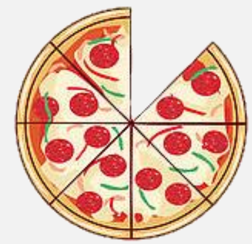
- There are several different areas inside of number theory.
  - Classical number theory (Prime numbers, divisors,…)
  - Analytic number theory (Use functions to study numbers)
  - Algebraic number theory (Use algebra to study numbers)
  - Diophantine geometry (Use geometry to study numbers)
  - ….

Numbers (数)

What types of numbers do you know?
Where/When do they appear?

1. Counting: x = number of people in this room $x = ?$

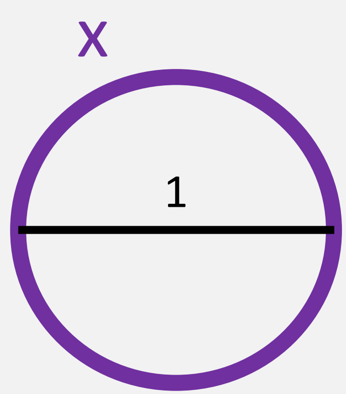
2. Ratios: x = remaining part of this pizza:



$x = ?$

3. Finding the root of a polynomial: $x^2 - 2 = 0$ $x = ?$
根 多項式

4. Circumference of a circle:
円周率



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Numbers (数)

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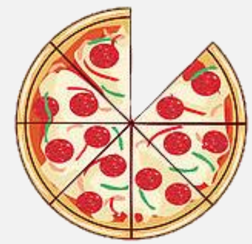


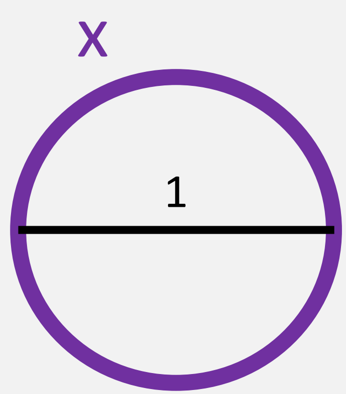
$$x = \frac{7}{8}$$

3. Finding the root of a polynomial:
根 多項式

$$x^2 - 2 = 0$$

$$x = \pm\sqrt{2}$$

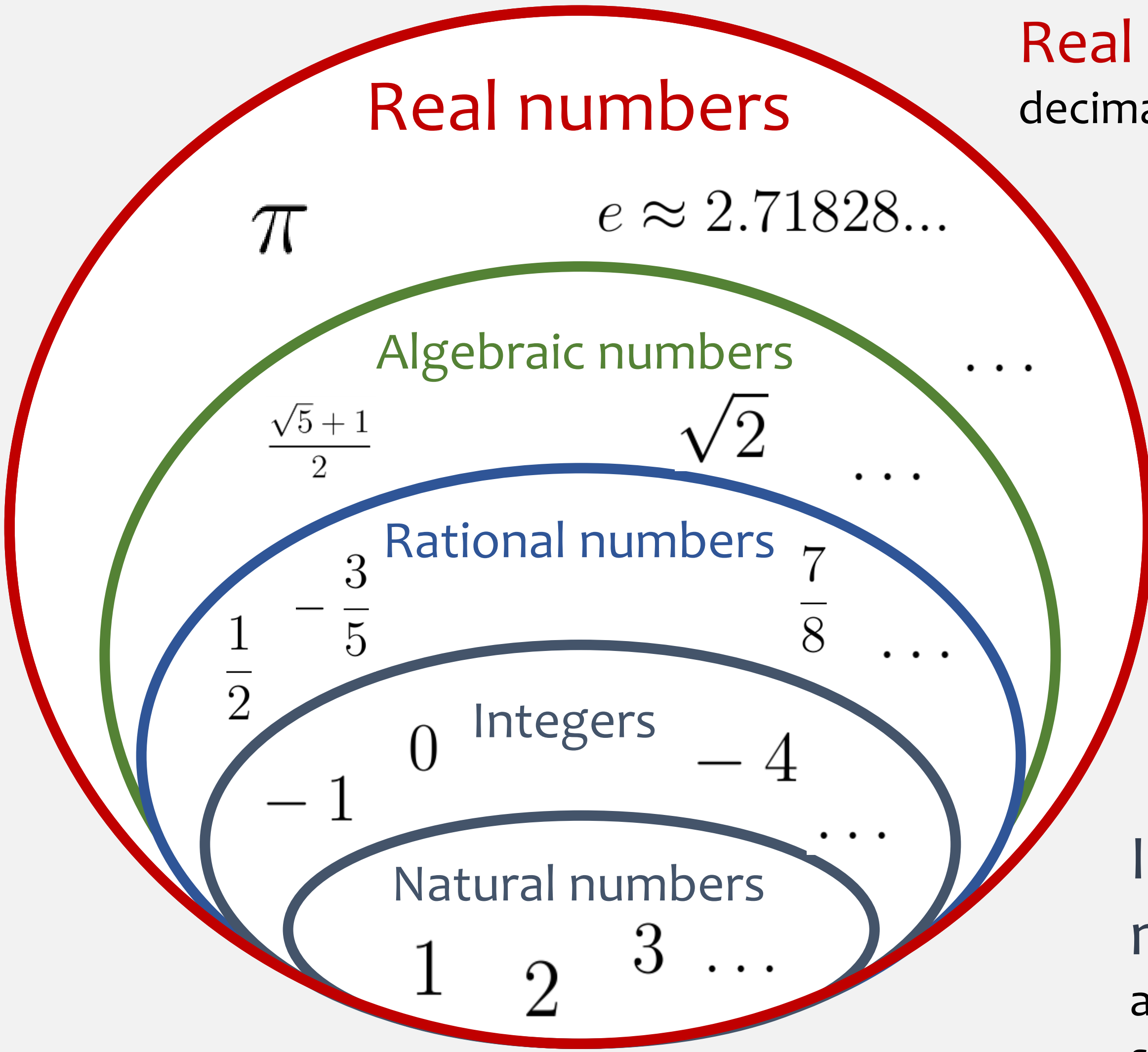
4. Circumference of a circle:
円周率



$$x = \pi$$

$$x \approx 3.1415926535\ldots$$

Overview of the real numbers (実数)



Real numbers: All numbers with a decimal expansion.

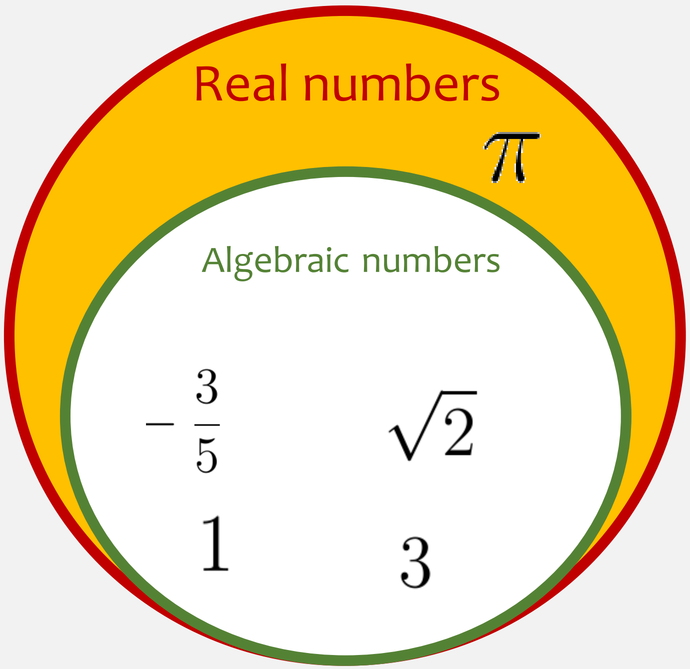
Algebraic numbers: Numbers which are zeros of Polynomials with rational coefficients.

Rational numbers: Numbers which can be written as a fraction $\frac{p}{q}$

Integers & Natural numbers: Numbers which arise from counting, adding and subtracting.

Transcendental numbers (超越数)

- Real numbers which are not algebraic are called **transcendental numbers**.



- There are **a lot of** transcendental numbers.

- In 1882 F. Lindemann proved that π is transcendental.

- This means you can **not** find a polynomial P(x) with π as its root, i.e. $P(\pi) = 0$.

Transcendental numbers (超越数)

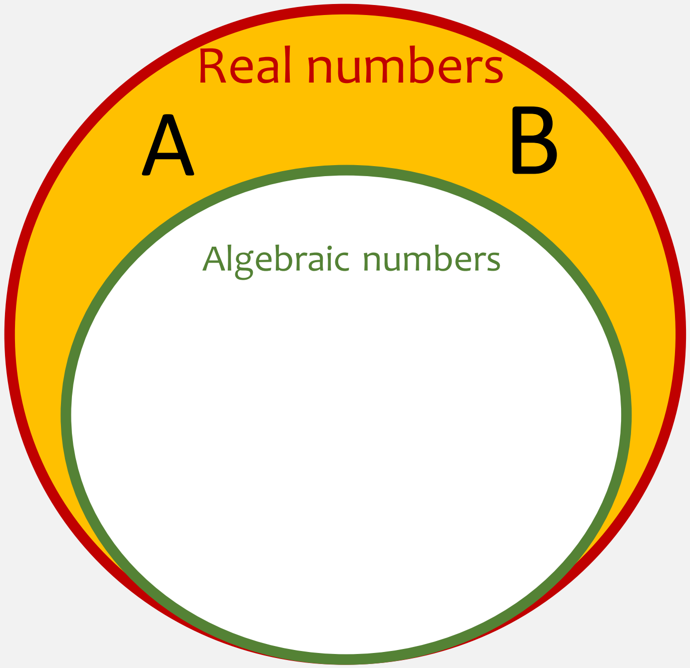
Exercise:

Suppose c is rational and
A and B are transcendental.

i) Is cA transcendental?

ii) Is A^2 transcendental?

iii) Is A+B transcendental?



Transcendental numbers (超越数)

- There are **a lot of numbers** where **we do not know** if they are algebraic or transcendental.

- To introduce some of them I first need to explain infinite series.

Finite sums & Infinite Series (無限級数)

- A series is a sum of numbers.
- The symbol $\sum$ is used for short notation

Example:

1) The sum of the first 100 natural numbers:

$$1 + 2 + 3 + 4 + \cdots + 100 = \sum_{n=1}^{100} n$$

2) The sum of the first six odd numbers

$$1 + 3 + 5 + 7 + 9 + 11 = \sum_{n=1}^{6} (2n - 1)$$

Finite sums & Infinite Series (無限級数)

In general if $a \leq b$ are integers and $f(n)$ is a term depending on n, we write

$$\sum_{n=a}^{b} f(n) = f(a) + f(a+1) + \cdots + f(b)$$

Exercise: Write the following sums with the symbol $\sum$

1) $$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$$

2) $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

Finite sums & Infinite Series (無限級数)

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Exercise: Write the following sums with the symbol $\sum$

1) $$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = \sum_{n=0}^{4} \frac{1}{2^n}$$

2) $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} = \sum_{n=1}^{5} \frac{1}{n^2}$$

Finite sums & Infinite Series (無限級数)

What happens if we calculate $\sum_{n=0}^{b} \frac{1}{2^n}$ for different b?

$$b = 0: \quad \sum_{n=0}^{0} \frac{1}{2^n} = 1$$

$$b = 1: \quad \sum_{n=0}^{1} \frac{1}{2^n} = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$b = 2: \quad \sum_{n=0}^{2} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} = 1.75$$

$$b = 3: \quad \sum_{n=0}^{3} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = 1.875$$

$$b = 10: \quad \sum_{n=0}^{10} \frac{1}{2^n} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{10}} = 1.9990234375$$

$$b = 100: \quad \sum_{n=0}^{100} \frac{1}{2^n} = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{100}} = 1.999999999999999999999999999992....$$

We see that this sum approaches 2 and we write $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$

This sum is called an **infinite Series**.

Finite sums & Infinite Series (無限級数)

In general one can proof (not hard!) the following:

For all real numbers q with $-1 < q < 1$ we have

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}\,.$$

So in the case $q = \frac{1}{2}$ this sum equals $\frac{1}{1-\frac{1}{2}} = 2$.

This sum is called the **geometric series**.

幾何級数

Finite sums & Infinite Series (無限級数)

But what happens if we calculate $\sum_{n=1}^{b} \frac{1}{n^2}$ for different b?

$$b = 1: \quad \sum_{n=1}^{1} \frac{1}{n^2} = 1$$

$$b = 2: \quad \sum_{n=1}^{2} \frac{1}{n^2} = 1 + \frac{1}{2^2} = 1.25$$

$$b = 3: \quad \sum_{n=1}^{3} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} = \frac{49}{36} = 1.361111....$$

$$b = 10: \quad \sum_{n=1}^{10} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{10^2} = 1.549767731166540690...$$

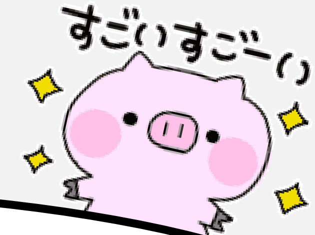
$$b = 100: \quad \sum_{n=1}^{100} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{100^2} = 1.634983900184892865...$$

$$b = 1000: \quad \sum_{n=1}^{1000} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{1000^2} = 1.643934566681559803...$$

What is $\sum_{n=1}^{\infty} \frac{1}{n^2}$??

Infinite Series (無限級数)

This problem was first solved by L. Euler in 1735



Leonhard Euler

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \approx 1.64493...$$

He gave a formula for all sums of this type with **even** exponents.

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots = \frac{\pi^6}{945}$$

Riemann zeta values (リーマンゼータ値)

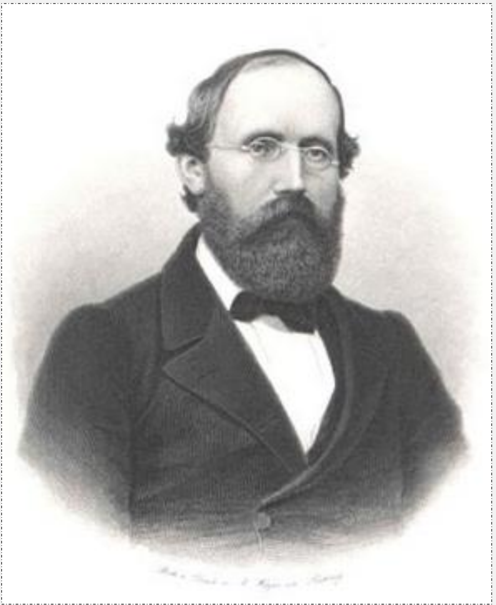
More general Euler considered for arbitrary k = 2,3,4,5,…
the numbers

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \dots$$

which are called **Riemann zeta values**.

$$\zeta(2) = \frac{\pi^2}{6}\,,\ \zeta(4) = \frac{\pi^5}{90}\,,\ \zeta(6) = \frac{\pi^6}{945}\,.$$

What about $\zeta(3)\,,\zeta(5)\,,\zeta(7),$ … ?

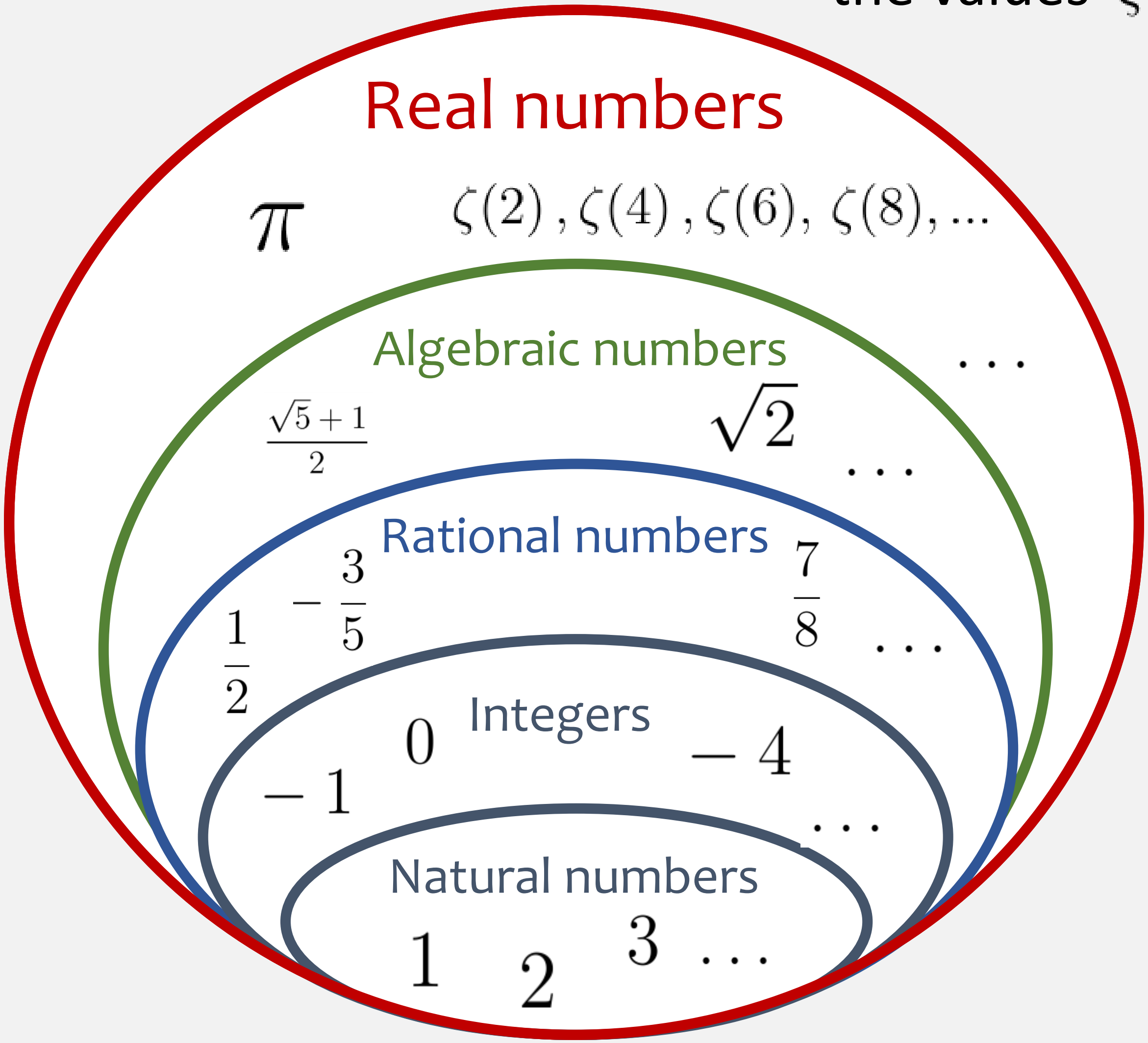


Bernhard Riemann

Riemann zeta values (リーマンゼータ値)

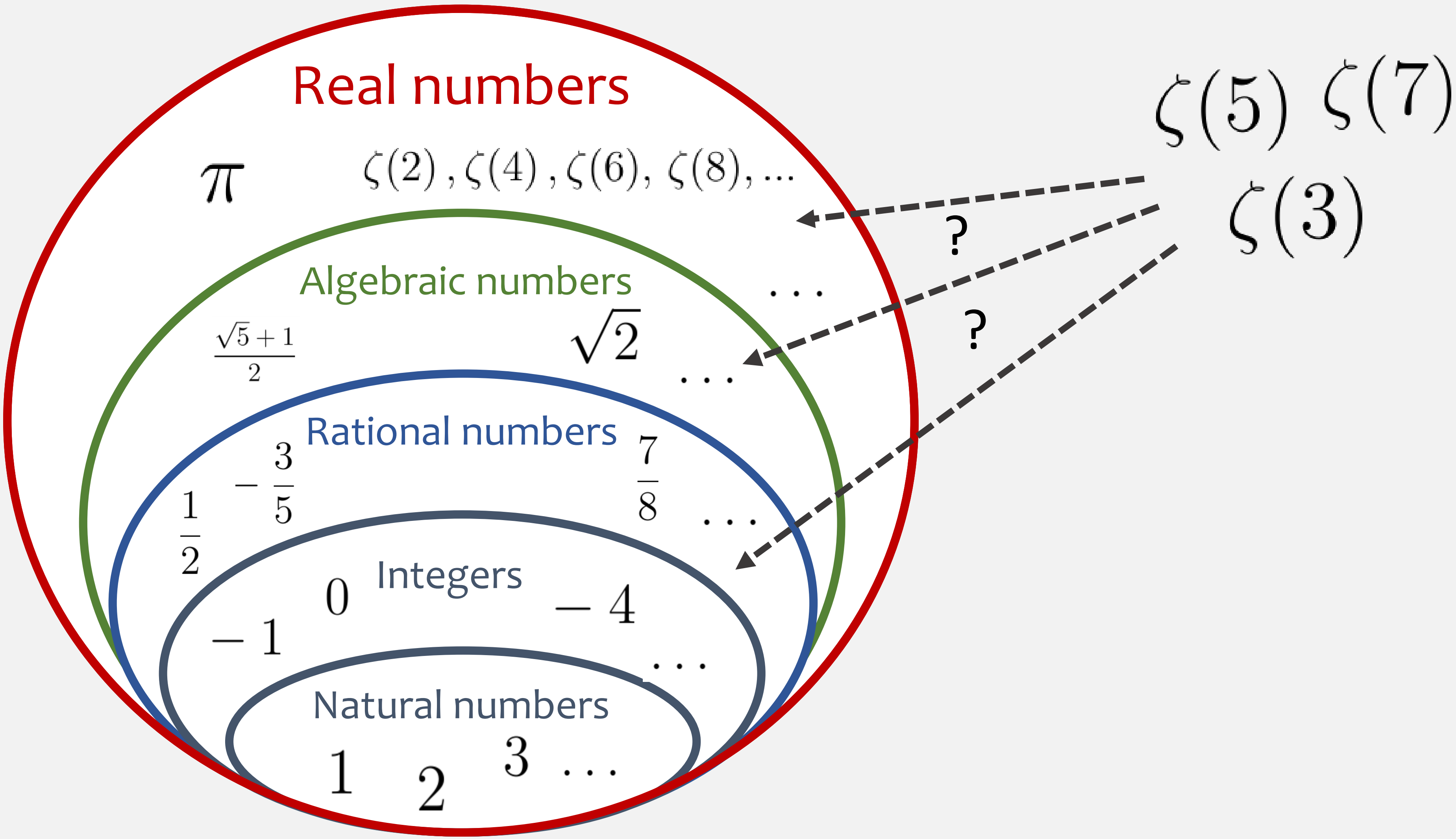
偶数

Due to Euler we know that for **even** k the values $\zeta(k)$ are transcendental.



Riemann zeta values (リーマンゼータ値)

For **odd** k nobody knows where to put $\zeta(k)$ in this picture....
奇数



Riemann zeta values (リーマンゼータ値)

For **odd** k nobody knows where to put $\zeta(k)$ in this picture….
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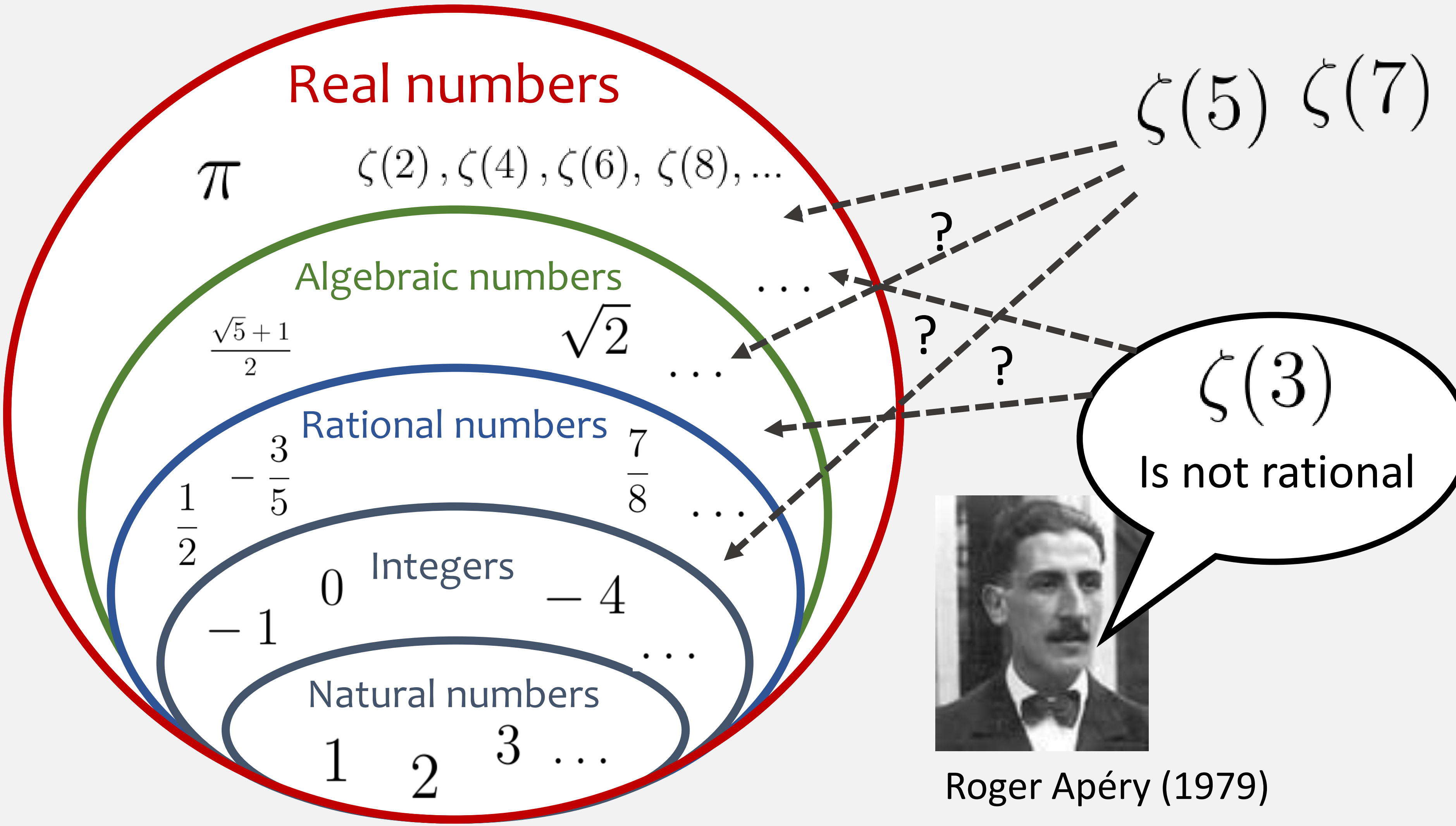


Roger Apéry (1979)

Multiple zeta values (多重ゼータ値)

- In my research I consider a generalization of the Riemann zeta values.

- For this we need to explain a new notation for infinite series:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \sum_{n>0} \frac{1}{n^k}$$

Sum over the numbers n=1,2,… = Sum over numbers n with n>0

Multiple zeta values (多重ゼータ値)

- The **double zeta values** are defined by

二重ゼータ値

$$\zeta(k_1, k_2) = \sum_{0<n_1<n_2} \frac{1}{n_1^{k_1} \cdot n_2^{k_2}} = \frac{1}{1^{k_1} \cdot 2^{k_2}} + \frac{1}{1^{k_1} \cdot 3^{k_2}} + \frac{1}{2^{k_1} \cdot 3^{k_2}} + \frac{1}{1^{k_1} \cdot 4^{k_2}} + \frac{1}{2^{k_1} \cdot 4^{k_2}} + \dots$$

0<1<2 0<1<3 0<2<3 0<1<4 0<2<4

Sum over all numbers n_1 and n_2 with $0<n_1 < n_2$

- The **multiple zeta values** are then defined by

$$\zeta(k_1, \dots, k_r) = \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

Multiple zeta values (多重ゼータ値)

- These numbers satisfy a lot of linear relations

Examples:

$$\zeta(3) = \zeta(1,2)$$

$$\frac{5197}{691}\zeta(12) = 168\zeta(7,5) + 150\zeta(5,7) + 28\zeta(3,9)$$

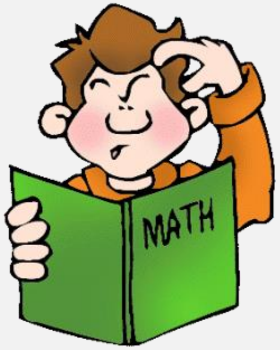
$$\zeta(\underbrace{2,...,2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}$$

- One of the goals is to understand all these relations

Mathematics

Why should you study Mathematics?

- It is fun!
- Japan is a good place to study mathematics!



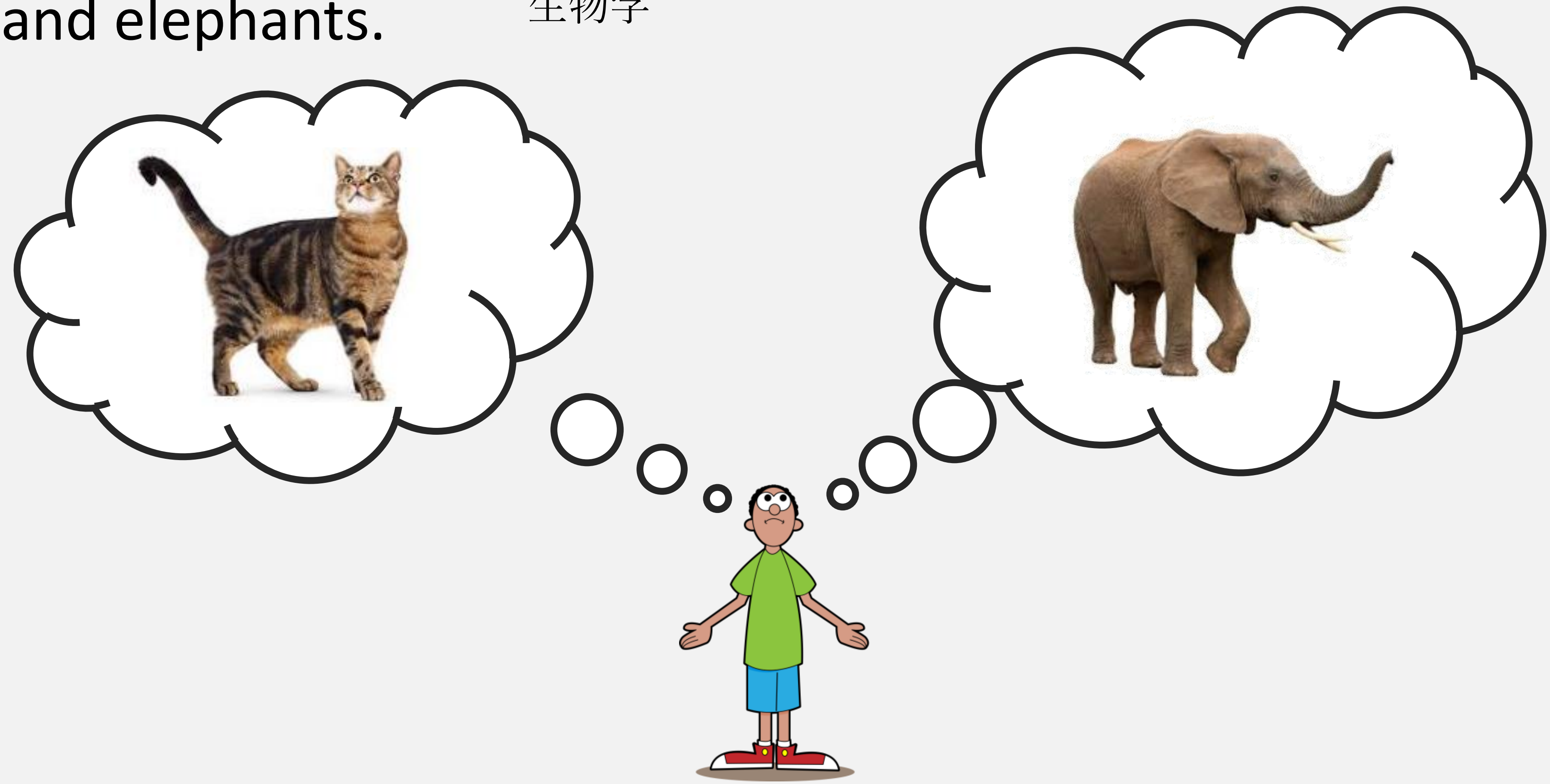
I like Mathematics because….

- …there are no limits.
- …there is just one “right” and “wrong”.
- …it is international.

Mathematics – No Limits

Imagine you study biology and you are interested in cats and elephants.

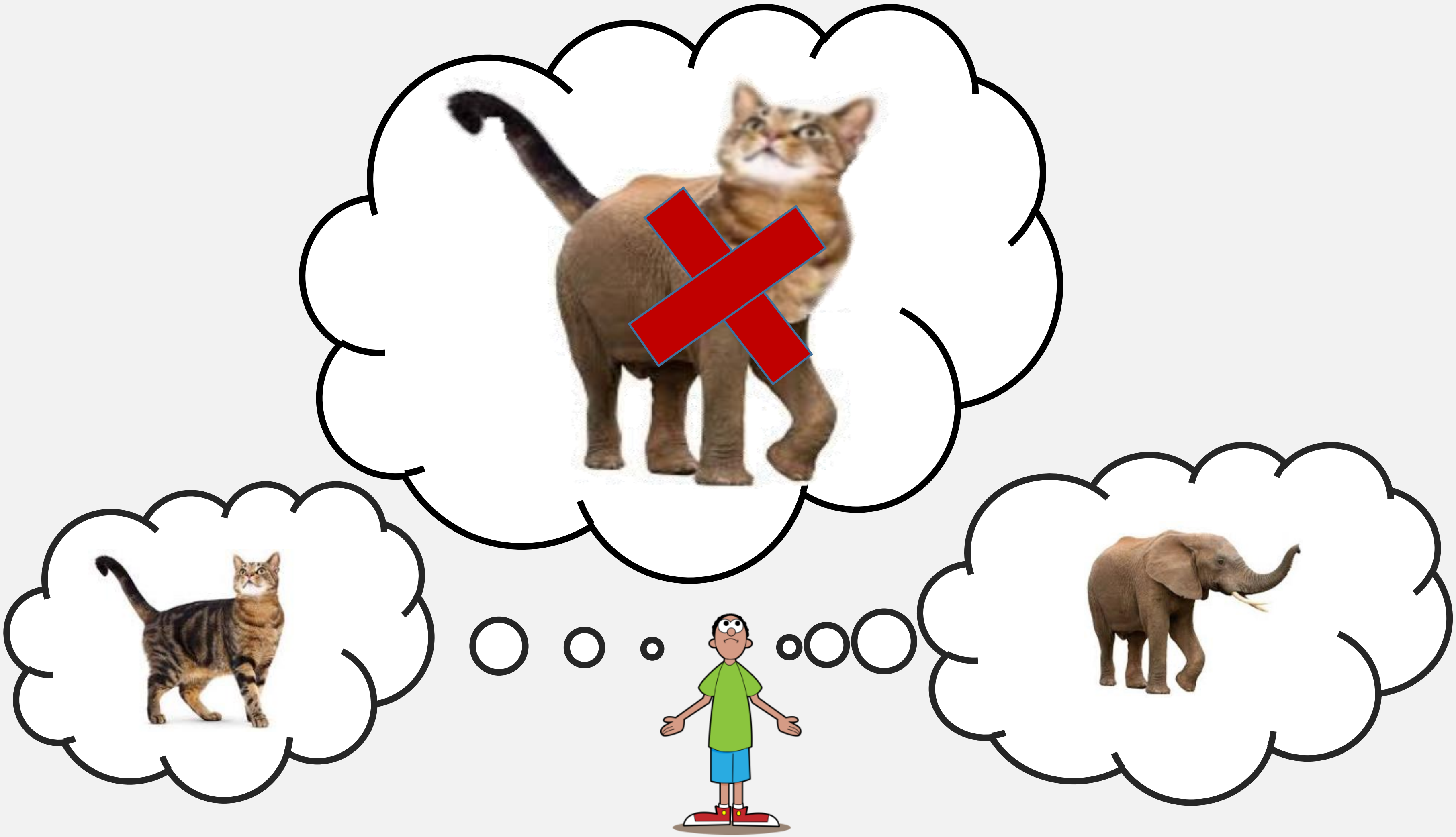
生物学



How to decide the topic for your thesis?

主題 学位論文

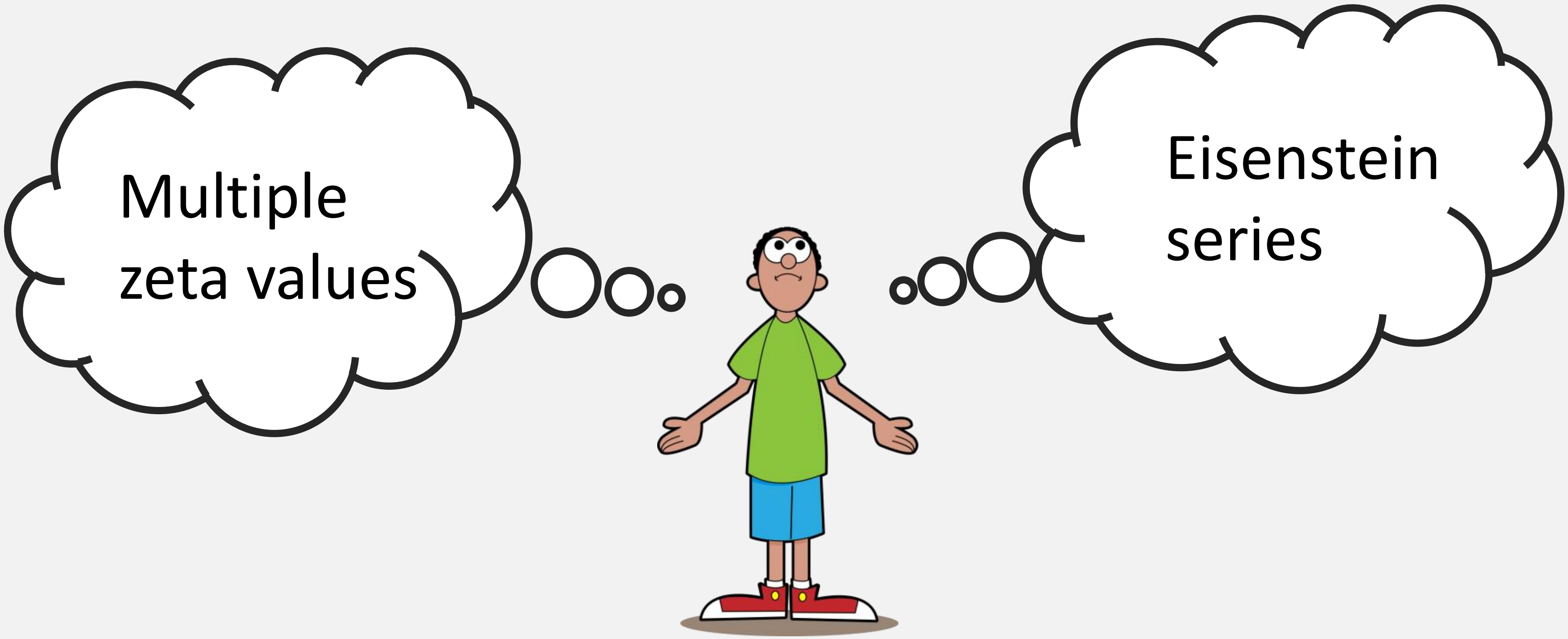
Mathematics – No Limits



You can not write a thesis on both
since there are no catephants!

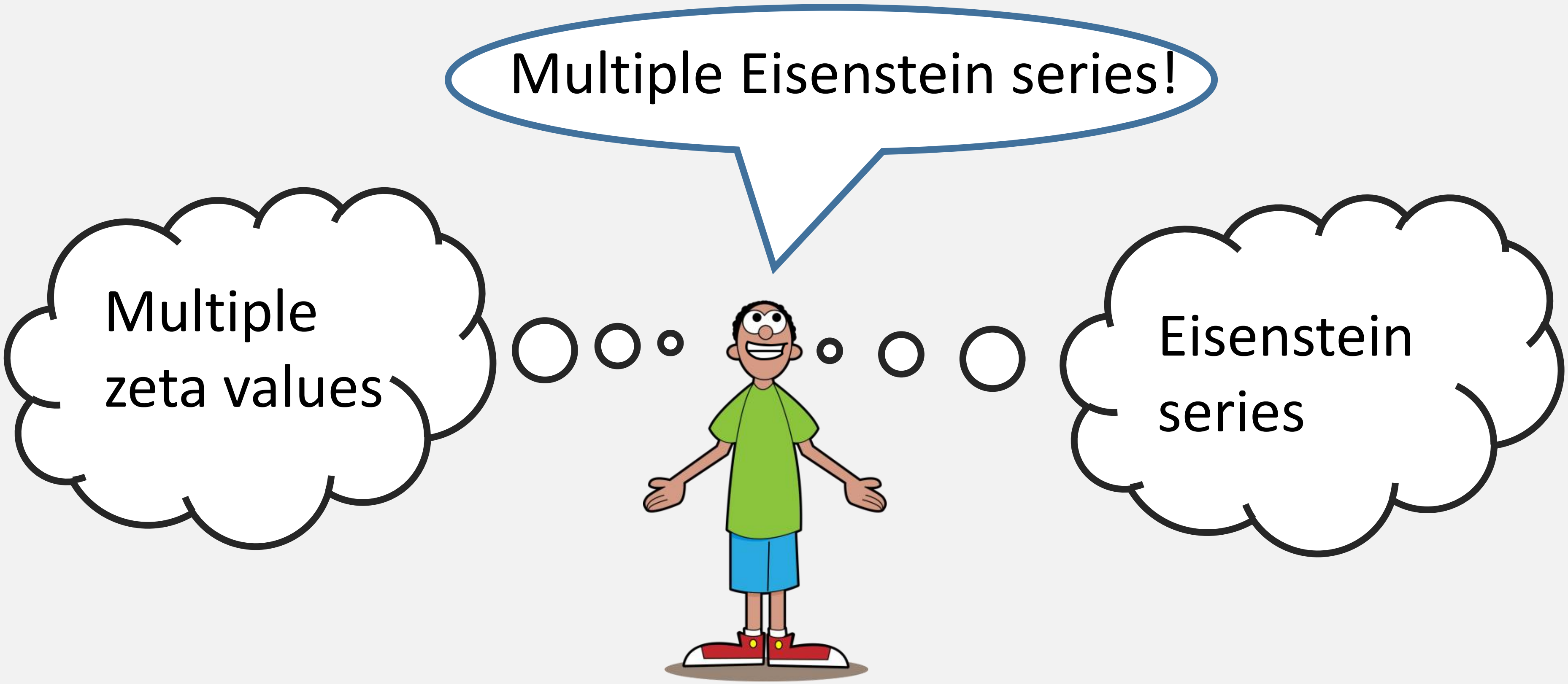
Mathematics – No Limits

- In Mathematics there are no such limits.
- I was also interested in two different topics (Multiple zeta values and Eisenstein series).



Mathematics – No Limits

My Master thesis & PhD Thesis was devoted to Multiple Eisenstein series which combined both topics.

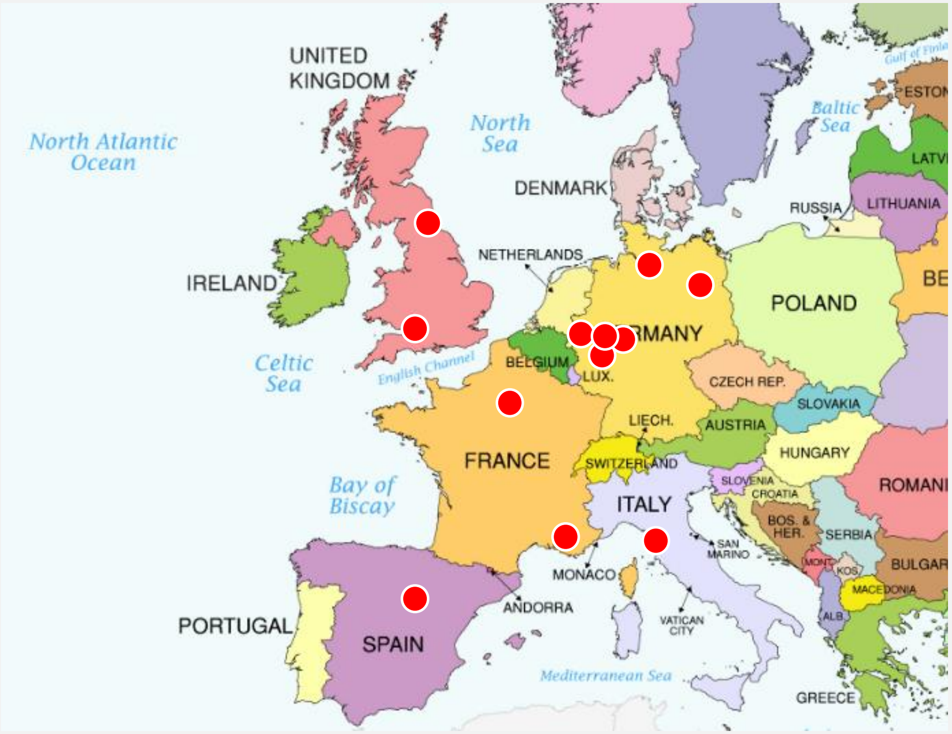


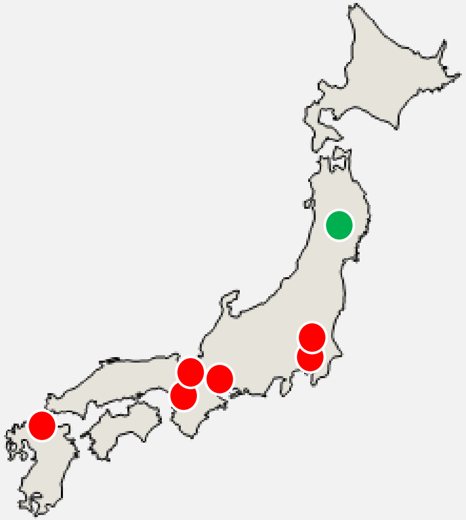
Mathematics – No Limits

- There are endless open problems in mathematics
- You can come up with your own questions and objects
- There is a lot of freedom since you do not need to think about an application before you start working on a problem.

Mathematics – International

- During my research I was able to travel a lot.
- I met a lot of people from different countries.





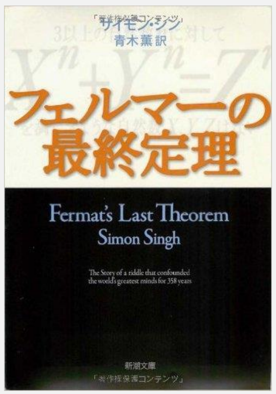
- Usually everybody in mathematical research can speak (simple) English! **LEARN ENGLISH!**

Where to start….?

- Study mathematics!
- Also try to read English textbooks or research papers!

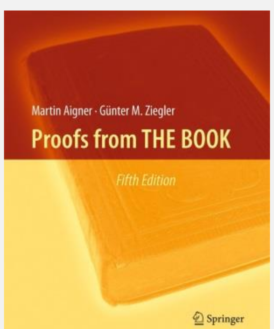
(www.arxiv.org)

Other nice books to start:



Fermat's Last Theorem - Simon Singh
Story about a nice mathematical problem, which was solved 1994.
This book is also available in Japanese.

And there is an English documentary (video) about this:
http://www.dailymotion.com/video/x3wrbsb



Proofs from THE BOOK - Martin Aigner, Günter M. Ziegler

A collection of beautiful mathematical problems and proofs.

Thank you for your attention!