Part 3: An analogue of Racinet's approach to formal multiple zeta values

3.1. Formal multiple zeta values (new)

- $\mathcal{X} = \{x_0, x_1\} \rightsquigarrow \mathfrak{h} = \mathbb{Q}\langle x_0, x_1\rangle$
- $\mathcal{Y} = \{y_1, y_2, \dots\}, \quad y_k \mathrel{\hat{=}} x_0^{k-1} x_1 \quad$ for $k \geq 1$.

$\boxed{\text{Reminder:}}$ The algebra of formal MZV is

$$\mathcal{Z} = \mathfrak{h}^1_{ш} / \mathrm{EDS}_{ш},$$

where EDS is the ideal in $\mathfrak{h}^1_{ш}$ generated by x_1 and $w * v - w ш v$ for $w \in \mathfrak{h}^0$, $v \in \mathfrak{h}^1$.

Applying again regularization, one obtains

$$\mathcal{Z}^f = \mathfrak{h}_{ш} / \widetilde{\mathrm{EDS}},$$

where $\widetilde{\mathrm{EDS}}$ is the ideal generated by x_0, x_1 and $w * v - w ш v$ for $w \in \mathfrak{h}_0$, $v \in \mathfrak{h}^1$.

Denote the image of $w \in \mathbb{Q}\langle \mathcal{X}\rangle$ in the quotient $\mathcal{Z}^f$ by $\mathcal{J}^f_{ш}(w)$.

3.2. Racinet's approach

- $\Delta_ш : \mathbb{Q}\langle X\rangle \to \mathbb{Q}\langle X\rangle \otimes \mathbb{Q}\langle X\rangle$ is the dual coproduct to ш, i.e.,
$$\Delta_ш(x_i) = x_i \otimes 1 + 1 \otimes x_i, \quad i \in \{0,1\}.$$

- $\Delta_* : \mathbb{Q}\langle Y\rangle \to \mathbb{Q}\langle Y\rangle \otimes \mathbb{Q}\langle Y\rangle$ is the dual coproduct to $*$, i.e.,
$$\Delta_*(y_i) = y_i \otimes 1 + 1 \otimes y_i + \sum_{j=1}^{i-1} y_j \otimes y_{i-j}, \quad i \geq 1.$$

$\boxed{\text{Definition:}}$ (Racinet)

For a commutative $\mathbb{Q}$-algebra R with unit, the set $DM(R)$ consists of all $\phi \in R\langle\langle X\rangle\rangle$ satisfying

coefficient of x_0 in ϕ

(i) $(\phi | x_0) = (\phi | x_1) = 0$

(ii) $\Delta_ш(\phi) = \phi \hat{\otimes} \phi$

(iii) $\Delta_*(\phi_*) = \phi_* \hat{\otimes} \phi_*$

connection between shuffle & stuffle regularization (see IkZ)

where $\phi_* = \exp\left(\sum_{n\geq 2} \frac{(-1)^{n-1}}{n} (\pi_y(\phi)|y_n) y_1^n\right) \pi_y(\phi) \in R\langle\langle Y\rangle\rangle.$

Let $DM_0(R)$ be the subset of $DM(R)$ consisting of all $\phi \in R\langle\langle X\rangle\rangle$ additionally satisfying

(iv) $(\phi | x_0 x_1) = 0$

The conditions (i)-(iii) essentially reformulate the relations in $\mathcal{Z}^f$. This leads to the following.

$\boxed{\text{Proposition:}}$ (Racinet)

$DM: \mathbb{Q}\text{-Alg} \to \text{Sets}$, $R \mapsto DM(R)$ is an affine scheme represented by $\mathcal{Z}^f$.

This means, there are bijections

$$DM(R) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}\text{-Alg}}(\mathcal{Z}^f, R).$$

Racinet also defined the corresponding linearized space $\mathfrak{dm}_0$ and showed that this is equipped with the Ihara bracket a Lie algebra.

Requiring some more work, one derives the following.

$\boxed{\text{Theorem:}}$ (Racinet)

$DM_0: \mathbb{Q}\text{-Alg} \to \text{Groups}$, $R \mapsto DM_0(R)$ is an affine group scheme (represented by $\mathcal{Z}^f/(\zeta^f(2))$).

[Group multiplication on $DM_0(R)$:

$$\Phi_1 \circledast \Phi_2 = \Phi_1(x_0, x_1) \cdot \Phi_2(x_0, \Phi^{-1} x_1 \Phi).$$
]

We deduce the following structural results for the algebra $\mathcal{Z}^f$.

$\boxed{\text{Corollary / Theorem:}}$ (Racinet, Goncharov)

(1) $\mathcal{Z}^f$ is a free polynomial algebra. More precisely, we have an isomorphism
$$\mathcal{Z}^f \cong \mathbb{Q}[\zeta^f(2)] \otimes \mathcal{U}(\partial m_0)^\vee.$$

(2) $\mathcal{Z}/(\zeta^f(2))$ equipped with Goncharov's coproduct is a Hopf algebra

dual to the group mult. on DM_0

To give an analog approach to the algebra $\mathcal{G}^f$, we need to regularize $\mathbb{Q}\langle\mathcal{B}\rangle^0$ to $\mathbb{Q}\langle\mathcal{B}\rangle$.

3.3 Regularization in $\mathbb{Q}\langle\mathcal{B}\rangle$

$\boxed{\text{Proposition:}}$ The map

formal variable

$$\mathrm{reg}_T : \mathbb{Q}\langle\mathcal{B}\rangle^0[T] \to \mathbb{Q}\langle\mathcal{B}\rangle,$$
$$\omega T^n \mapsto \omega *_b b_0^{*_b n},$$

is an algebra isomorphism for $*_b$.

This is very similar to the shuffle and stuffle regularization of MZV.

Applying first $\mathrm{reg}_T^{-1} : \mathbb{Q}\langle\mathcal{B}\rangle \to \mathbb{Q}\langle\mathcal{B}\rangle^0[T]$ and then evaluating in $T=0$ gives the regularization morphism

$$\mathrm{reg} : (\mathbb{Q}\langle\mathcal{B}\rangle, *_b) \longrightarrow (\mathbb{Q}\langle\mathcal{B}\rangle^0, *_b).$$

satisfying $\mathrm{reg}(b_0) = 0$

It is the unique algebra morphism, which extends the identity and satisfies this condition.

$\boxed{\text{Proposition:}}$ (B.) On generating series of words, we have

$$\mathrm{reg}\, \rho_{\mathcal{B}}(\mathcal{W})\binom{X_1, \dots, X_r}{Y_0; Y_1, \dots, Y_r} = \rho_{\mathcal{B}}^0(\mathcal{W}^0)\binom{X_1, \dots, X_r}{Y_1 - Y_0, \dots, Y_r - Y_0}.$$

$\boxed{\text{Theorem:}}$ (B.) We have

$$\mathfrak{g}^f \cong (\mathbb{Q}\langle\mathcal{B}\rangle, *_b)/\mathrm{Rel}_{\tau,0},$$

where $\mathrm{Rel}_{\tau,0}$ is the ideal generated by b_0 and $\omega - \tau(\omega)$ for $\omega \in \mathbb{Q}\langle \mathcal{B} \rangle^0$.

3.3. An affine scheme associated to $\mathcal{G}^f$

Let $\Delta_b : \mathbb{Q}\langle\mathcal{B}\rangle \to \mathbb{Q}\langle\mathcal{B}\rangle$ be the dual coproduct to $*_b$, i.e.,

$$\Delta_b(b_i) = b_i \otimes 1 + 1 \otimes b_i + \sum_{j=1}^{i-1} b_j \otimes b_{i-j}, \quad i \geq 0.$$

Again, we see that Δ_b is a combination of the dual coproducts $\Delta_ш$, Δ_*.
The coproduct does not preserve $\mathbb{Q}\langle\mathcal{B}\rangle^0$, this is why we gave the regularization.

$\boxed{\text{Definition:}}$ (B.) For each commutative $\mathbb{Q}$-algebra R with unit, let $BM(R)$ the set of all $\Phi \in R\langle\langle\mathcal{B}\rangle\rangle$ satisfying

$$\begin{aligned} &\text{(i)} & (\Phi \mid b_0) &= 0 \\ &\text{(ii)} & \Delta_b(\Phi) &= \Phi \hat{\otimes} \Phi \\ &\text{(iii)} & \tau(\pi_0(\Phi)) &= \pi_0(\Phi) \end{aligned}$$

where π_0 is the R-linear extension of the canonical projection

$$\mathbb{Q}\langle\mathcal{B}\rangle \to \mathbb{Q}\langle\mathcal{B}\rangle^0,$$

$$b_{s_1} \cdots b_{s_l} \mapsto \begin{cases} b_{s_1} \cdots b_{s_l}, & s_1 \geq 1 \\ 0 & \text{else} \end{cases}.$$

Let $BM_0(R)$ the subset of $BM(R)$ consisting of all $\Phi \in R\langle\langle\mathcal{B}\rangle\rangle$ additionally satisfying

$$\text{(iv)} \quad (\Phi | b_k) = 0 \qquad \text{for } k = 2, 4, 6.$$

The conditions (i)–(iii) essentially reformulate the relations in $\mathcal{G}^f$. This leads to the following.

$\boxed{\text{Theorem:}}$ (B.)

$BM: \mathbb{Q}\text{-Alg} \to \text{Sets}$, $R \mapsto BM(R)$ is an affine scheme represented by $\mathcal{G}^f$.

This means, there are bijections

$$BM(R) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}\text{-Alg}}(\mathcal{G}^f, R).$$

Consider the following Hopf algebra (anti) morphisms

$$\theta_x^{anti}: (R\langle\langle X\rangle\rangle, \text{conc}, \Delta_{ш}) \hookrightarrow (R\langle\langle\mathcal{B}\rangle\rangle, \text{conc}, \Delta_b),$$

$$x_{\varepsilon_1} \cdots x_{\varepsilon_n} \longmapsto b_{\varepsilon_n} \cdots b_{\varepsilon_1} \qquad (\varepsilon_i \in \{0,1\})$$

$$\theta_y : (R\langle\langle Y\rangle\rangle, \text{conc}, \Delta_*) \hookrightarrow (R\langle\langle\mathcal{B}\rangle\rangle, \text{conc}, \Delta_b)$$

$$y_{k_1} \cdots y_{k_r} \longmapsto b_{k_1} \cdots b_{k_r} \qquad (k_i \geq 1)$$

$\boxed{\text{Theorem:}}$ (B.) We have injective maps

$$\begin{aligned} \theta : \mathrm{DM}(R) &\longrightarrow \mathrm{BM}(R), \\ \phi &\longmapsto \theta_x^{\mathrm{anti}}(\phi)\,\theta_y(\phi_*). \end{aligned}$$

given in the definition of DM(R)

A key observation for the proof is that

$$\tau \circ \pi_0 \circ \theta_x^{\mathrm{anti}} = \theta_y \circ \pi_y .$$

In particular, this theorem shows that BM(R) is non-empty.

Applying Yoneda's lemma to the natural transformation $\theta : \mathrm{DM} \hookrightarrow \mathrm{BM}$ yields the following.

$\boxed{\text{Corollary:}}$ (B.) There is a surjective algebra morphism

$b_{\varepsilon_1} \cdots b_{\varepsilon_n} \mapsto x_{\varepsilon_n} \cdots x_{\varepsilon_1}$ $(\varepsilon_i \in \{0,1\})$

$b_{k_1} \cdots b_{k_r} \mapsto y_{k_1} \cdots y_{k_r}$ $(k_i \geq 1)$

$$\begin{aligned} \rho : \mathcal{G}^f &\longrightarrow \mathcal{Z}^f \\ G^f(\omega) &\longmapsto \sum_{\substack{\omega = uv \\ u \in \mathbb{Q}\langle b_0, b_1\rangle,\ v \in \mathbb{Q}\langle b_i \mid i \geq 1\rangle}} \zeta^f_{ш}\big((\theta_x^{\mathrm{anti}})^{-1}(u)\big)\, \zeta^f_{*}\big(\theta_y^{-1}(v)\big) \end{aligned}$$

So $\mathcal{G}^f$ is also a generalization of $\mathcal{Z}^f$. This map should be seen as a formal version of taking the limit $q \to 1$ or taking the constant term of q-series.

3.4. Outlook

There is also a linearized space $\mathfrak{bm}_0$.
In my thesis, I obtained a Lie bracket on $\mathbb{Q}\langle\mathcal{B}\rangle$, which generalizes the Ihara bracket on $\mathfrak{dm}_0$ and which conjecturally preserves $\mathfrak{bm}_0$. Therefore, we expect that BM_0 is an affine group scheme.
This leads to the following conjectured structures for $\mathcal{G}^f$.

Conjecture:

(1) $\mathcal{G}^f$ is a free polynomial algebra. More precisely, there is an algebra isomorphism

$$\mathcal{G}^f \cong \tilde{\mathcal{M}}(SL_2(\mathbb{Z})) \otimes \mathcal{U}(\mathfrak{bm}_0)^\vee$$

quasimodular forms

(2) $\mathcal{G}^f / (G^f(2), G^f(4), G^f(6))$ is a Hopf algebra, where the coproduct is a generalization of Goncharov's coproduct.