Part 2: The balanced setup

2.1. Second definition of $\mathcal{G}^f$

Aim: Give a definition of $\mathcal{G}^f$ in terms of the balanced quasi-shuffle algebra, and show that it is isomorphic to the description given in Part 1.

$\boxed{\text{Definition:}}$ Consider the alphabet $\mathcal{B} = \{b_0, b_1, b_2, \dots\}$, and denote by $\mathbb{Q}\langle\mathcal{B}\rangle$ the free non-commutative $\mathbb{Q}$-algebra generated by $\mathcal{B}$.
The balanced quasi-shuffle product $*_b$ on $\mathbb{Q}\langle\mathcal{B}\rangle$ is given by $1 *_b w = w *_b 1 = w$ and

$$b_i u *_b b_j v = b_i(u *_b b_j v) + b_j(b_i u * v) + \delta_{ij>0}\, b_{i+j}(u *_b v)$$

for all $u, v, w \in \mathbb{Q}\langle\mathcal{B}\rangle$

It is a combination of the stuffle product ($ij>0$) and the shuffle product ($i,j \in \{0,1\}$) of the multiple zeta values.

The balanced quasi-shuffle product is graded w.r.t. the weight,

$$\mathrm{wt}(b_{s_1} \cdots b_{s_l}) = s_1 + \dots + s_l + \#\{i \mid s_i = 0\}.$$

Denote $\mathbb{Q}\langle\mathcal{B}\rangle^\circ = \mathbb{Q}\langle\mathcal{B}\rangle \setminus b_0\,\mathbb{Q}\langle\mathcal{B}\rangle$, so $\mathbb{Q}\langle\mathcal{B}\rangle^\circ$ is spanned by all words not starting with b_0.

Definition: Let $\tau : \mathbb{Q}\langle\mathcal{B}\rangle^\circ \to \mathbb{Q}\langle\mathcal{B}\rangle^\circ$ be the involution given by

$$\tau(b_{k_1} b_0^{m_1} \cdots b_{k_r} b_0^{m_r}) = b_{m_r+1} b_0^{k_r-1} \cdots b_{m_1+1} b_0^{k_1-1}$$

for $k_1, \ldots, k_r \geq 1$, $m_1, \ldots, m_r \geq 0$.

Now, we are able to define $\mathcal{G}^f$ in terms of the balanced quasi-shuffle algebra.

Definition: We set

$$\mathcal{G}^f = (\mathbb{Q}\langle\mathcal{B}\rangle^\circ, \underset{\text{preserves } \mathbb{Q}\langle\mathcal{B}\rangle^\circ}{*_b}) \big/ \langle w - \tau(w) \mid w \in \mathbb{Q}\langle\mathcal{B}\rangle^\circ \rangle$$

Note that τ is much more easier to handle than the operator σ.

We will show that this definition is equivalent to the previous one by using generating series. Therefore, we will first discuss generating series associated to quasi-shuffle algebras in general.

2.2. Quasi-shuffle products & generating series

- alphabet $\mathcal{A}$: countable set
- $\mathcal{A}^* = \{a_1 \cdots a_n \mid a_i \in \mathcal{A}\} \cup \{1\}$
- $\mathbb{Q}\mathcal{A}$: $\mathbb{Q}$-vector space with basis $\mathcal{A}$
- $\mathbb{Q}\langle\mathcal{A}\rangle$: free, non-commutative $\mathbb{Q}$-algebra generated by $\mathcal{A}$

$\boxed{\text{Definition:}}$ Let $\diamond : \mathbb{Q}\mathcal{A} \times \mathbb{Q}\mathcal{A} \to \mathbb{Q}\mathcal{A}$ be an associative, commutative product. Then the product $*_\diamond$ on $\mathbb{Q}\langle\mathcal{A}\rangle$ is recursively defined by $1 *_\diamond w = w *_\diamond 1 = w$ and

$$au *_\diamond bv = a(u *_\diamond bv) + b(au *_\diamond v) + (a \diamond b)(u *_\diamond v)$$

for all $u, v, w \in \mathbb{Q}\langle\mathcal{A}\rangle$.

$\boxed{\text{Theorem:}}$ (Hoffman)

$(\mathbb{Q}\langle\mathcal{A}\rangle, *_\diamond)$ is an associative, commutative algebra.

In the following, let $(\mathbb{Q}\langle\mathcal{A}\rangle, *_\diamond)$ be any fixed quasi-shuffle algebra.

Define the generic diagonal series of $\mathbb{Q}\langle\mathcal{A}\rangle$ by

$$\mathfrak{W}(\mathcal{A}) = \sum_{w \in \mathcal{A}^*} w \otimes w.$$

We want to apply maps to the first and the second tensor product factors, to obtain different kinds of generating series.

Want to explain the following picture:

$$
\begin{tikzcd}
& \mathcal{W}(\mathcal{A}) \arrow[dl, "\varphi\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\rho"] & \\
\sum_{w\in\mathcal{A}^*}\varphi(w)w \arrow[dr, "\mathrm{id}\otimes\rho"'] & & \left(\sum_{w\in(\mathcal{A}^*)^{(r)}} w\rho(w)\right)_{r\geq 0} \arrow[dl, "\varphi\otimes\mathrm{id}"] \\
& \left(\sum_{w\in(\mathcal{A}^*)^{(r)}} \varphi(w)\rho(w)\right)_{r\geq 0} &
\end{tikzcd}
$$

"non-commutative generating series"

"generating series of words"

ordered by depth

"(bi-) moulds, i.e. commutative generating series"

Let $\mathrm{dep}: \mathcal{A}^* \to \mathbb{Z}_{\geq 0}$ be a depth map compatible with concatenation, i.e.,

$$\mathrm{dep}(uv) = \mathrm{dep}(u) + \mathrm{dep}(v)$$

for all $u, v \in \mathcal{A}^*$.

We denote:

- $(\mathcal{A}^*)^{(r)}$ = set of all words in $\mathcal{A}^*$ of depth r,
- $\mathbb{Q}\langle\mathcal{A}\rangle^{(r)}$ = subspace of $\mathbb{Q}\langle\mathcal{A}\rangle$ spanned by all words of depth r.

Step 1: The maps ρ

$\boxed{\text{Definition:}}$ Let $\rho_{\mathcal{A}} : \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}[z_1, z_2, \dots]$ be a $\mathbb{Q}$-linear map with the following properties:

(i) There is a strictly increasing sequence

$$\ell(0) < \ell(1) < \ell(2) < \dots$$

of non-negative integers, s.th.

$$\rho_{\mathcal{A}}\big|_{\mathbb{Q}\langle\mathcal{A}\rangle^{(r)}} : \mathbb{Q}\langle\mathcal{A}\rangle^{(r)} \longrightarrow \mathbb{Q}[z_1, \dots, z_{\ell(r)}] \qquad \left(\begin{array}{l}\ell(0)=0 \\ \Rightarrow \rho_{\mathcal{A}}(\mathbb{Q}\langle\mathcal{A}\rangle^{(0)}) \subset \mathbb{Q}\end{array}\right)$$

is injective.

shifting the variables z_i to $z_{\ell(r)+i}$,
so $\rho_{\mathcal{A}}^{[\ell(r)]}(\mathbb{Q}\langle\mathcal{A}\rangle^{(n)}) \subset \mathbb{Q}[z_{\ell(r)+1}, \dots, z_{\ell(r)+\ell(n)}]$

(ii) We have

$$\rho_{\mathcal{A}}(uv) = \rho_{\mathcal{A}}(u)\, \rho_{\mathcal{A}}^{[\ell(r)]}(v)$$

for $u \in \mathbb{Q}\langle\mathcal{A}\rangle^{(r)}$, $v \in \mathbb{Q}\langle\mathcal{A}\rangle$.

The (commutative) generating series of words in $\mathbb{Q}\langle\mathcal{A}\rangle$ associated to $\rho_{\mathcal{A}}$ are

$$\rho_{\mathcal{A}}(\mathcal{W})_r(z_1, \dots, z_{\ell(r)}) = \sum_{w \in (\mathcal{A}^*)^{(r)}} w\, \rho(w) \in \mathbb{Q}\langle\mathcal{A}\rangle[[z_1, \dots, z_{\ell(r)}]]$$

for $r \geq 0$.

for simplicity, we will often
drop this index

Example:

Consider the quasi-shuffle algebra $(\mathbb{Q}\langle\mathcal{A}^{bi}\rangle, *)$

with

- $\mathcal{A}^{bi} = \{ \left[\begin{smallmatrix} k \\ d \end{smallmatrix}\right] \mid k \geq 1, d \geq 0 \}$,
- $au * bv = a(u * bv) + b(au * v) + a \diamond b (u * v)$
  and $\left[\begin{smallmatrix} k_1 \\ d_1 \end{smallmatrix}\right] \diamond \left[\begin{smallmatrix} k_2 \\ d_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} k_1+k_2 \\ d_1+d_2 \end{smallmatrix}\right]$,
- $\operatorname{dep}\left(\left[\begin{smallmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{smallmatrix}\right]\right) = r$.

Define the $\mathbb{Q}$-linear map

$$\rho_{\mathcal{A}^{bi}} : \mathbb{Q}\langle \mathcal{A}^{bi} \rangle \to \mathbb{Q}[X_1, Y_1, X_2, Y_2, \dots],$$
$$\left[\begin{smallmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{smallmatrix}\right] \mapsto X_1^{k_1-1} \frac{Y_1^{d_1}}{d_1!} \cdots X_r^{k_r-1} \frac{Y_r^{d_r}}{d_r!}.$$

One checks that it satisfies the previously given conditions (with $\ell(r) = 2r$).

We get $\rho_{\mathcal{A}^{bi}}(\mathcal{W})_0 = 1$ and

$$\underbrace{\rho_{\mathcal{A}^{bi}}(\mathcal{W})_r}_{= \mathcal{A} \text{ (in Henrik's talk)}} \binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{\substack{k_1, \dots, k_r \geq 1 \\ d_1, \dots, d_r \geq 0}} \left[\begin{smallmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{smallmatrix}\right] X_1^{k_1-1} \frac{Y_1^{d_1}}{d_1!} \cdots X_r^{k_r-1} \frac{Y_r^{d_r}}{d_r!}.$$

for $r \geq 1$.

$\boxed{\text{Proposition:}}$ Let $\rho_{\mathcal{A}} : \mathbb{Q}\langle \mathcal{A} \rangle \to \mathbb{Q}[Z_1, Z_2, \dots]$ as before with

$$\ell(r_1) + \ell(r_2) = \ell(r_1 + r_2), \quad r_1, r_2 \geq 0.$$

Then, we have for all $0 \leq n \leq r$

$$\rho_{\mathcal{A}}(\mathcal{W})_n (Z_1, \dots, Z_{\ell(n)}) \cdot \rho_{\mathcal{A}}(\mathcal{W})_{r-n} (Z_{\ell(n)+1}, \dots, Z_{\ell(r)})$$

concatenation product $\mathbb{Q}[[Z_1, Z_2, \dots]]$-linearly extended to $\mathbb{Q}\langle \mathcal{A} \rangle [[Z_1, Z_2, \dots]]$

$$= \rho_{\mathcal{A}}(\mathcal{W})_r (Z_1, \dots, Z_{\ell(r)}).$$

Proof: For $0<n<r$ compute

$$\rho_{\mathcal{A}}(\mathcal{W})(z_1,\dots,z_{e(n)})\cdot\rho_{\mathcal{A}}(\mathcal{W})(z_{e(n)+1},\dots,z_{e(r)})$$

$$= \sum_{u\in(\mathcal{A}^*)^{(n)}}\sum_{v\in(\mathcal{A}^*)^{(r-n)}} uv\,\rho_{\mathcal{A}}(u)\,\rho_{\mathcal{A}}^{[e(n)]}(v)$$

$$\overset{(ii)}{=} \sum_{u\in(\mathcal{A}^*)^{(n)}}\sum_{v\in(\mathcal{A}^*)^{(r-n)}} uv\,\rho_{\mathcal{A}}(uv)$$

$$\overset{\omega=uv}{=} \sum_{\omega\in(\mathcal{A}^*)^{(r)}} \omega\,\rho_{\mathcal{A}}(\omega)$$

$$= \rho_{\mathcal{A}}(\mathcal{W})(z_1,\dots,z_{e(r)}).\qquad\square$$

Some well-behaved quasi-shuffle product can be also described on the generating series of words by an explicit recursive formula with respect to concatenation. For example, for the shuffle and stuffle product of multiple zeta values, this was done by K. Ihara.

$\boxed{\text{Theorem:}}$ For the product $*$ on $\mathbb{Q}\langle\mathcal{A}^{bi}\rangle$ one obtains for $0<n<r$ that

$$1*\rho_{\mathcal{A}^{bi}}(\mathcal{W})_n = \rho_{\mathcal{A}^{bi}}(\mathcal{W})_n * 1 = \rho_{\mathcal{A}^{bi}}(\mathcal{W})_n\,,$$

$$\rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_1,\dots,x_n}{y_1,\dots,y_n} * \rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_{n+1},\dots,x_r}{y_{n+1},\dots,y_r}$$
$$= \rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_1}{y_1}\cdot\left(\rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_2,\dots,x_n}{y_2,\dots,y_n} * \rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_{n+1},\dots,x_r}{y_{n+1},\dots,y_r}\right)$$
$$+ \rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_{n+1}}{y_{n+1}}\cdot\left(\rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_1,\dots,x_n}{y_1,\dots,y_n} * \rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_{n+2},\dots,x_r}{y_{n+2},\dots,y_r}\right)$$
$$+ \frac{\rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_1}{y_1+y_{n+1}} - \rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_{n+1}}{y_1+y_{n+1}}}{x_1 - x_{n+1}} \cdot\left(\rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_2,\dots,x_n}{y_2,\dots,y_n} * \rho_{\mathcal{A}}^{bi}(\mathcal{W})\binom{x_{n+2},\dots,x_r}{y_{n+2},\dots,y_r}\right).$$

Proof: straight-forward inductive calculations.

Consider the alphabet $\mathcal{B} = \{b_0, b_1, b_2, \dots\}$ and set

$$\operatorname{dep}(b_{s_1}\cdots b_{s_\ell}) = \ell - \#\{i \mid s_i = 0\}.$$

Define the $\mathbb{Q}$-linear map

$$\rho_{\mathcal{B}}: \mathbb{Q}\langle\mathcal{B}\rangle \to \mathbb{Q}[Y_0, X_1, Y_1, X_2, Y_2, \dots]$$
$$b_0^{m_0} b_{k_1} b_0^{m_1} \cdots b_{k_r} b_0^{m_r} \mapsto Y_0^{m_0} X_1^{k_1-1} Y_1^{m_1} \cdots X_r^{k_r-1} Y_r^{m_r}$$

Then $\rho_{\mathcal{B}}$ satisfies the previously given conditions (with $\ell(r) = 2r+1$).

We get for $r \geq 0$,

$$\rho_{\mathcal{A}}(\mathcal{W})_r\binom{x_1,\dots,x_r}{y_0; y_1,\dots,y_r} = \sum_{\substack{k_1,\dots,k_r \geq 1\\ m_0,\dots,m_r \geq 0}} b_0^{m_0} b_{k_1} b_0^{m_1} \cdots b_{k_r} b_0^{m_r}\, Y_0^{m_0} X_1^{k_1-1} Y_1^{m_1} \cdots X_r^{k_r-1} Y_r^{m_r}.$$

For the definition of g^f, we used the subspace $\mathbb{Q}\langle\mathcal{B}\rangle^0$. Therefore, we define also the following.

Define
$$\mathcal{W}(\mathcal{B})^0 = \sum_{w \in \mathcal{B}^* \setminus b_0\mathcal{B}^*} w \otimes w .$$

Let $\rho^0_{\mathcal{B}} : \mathbb{Q}\langle\mathcal{B}\rangle^0 \to \mathbb{Q}[X_1, Y_1, X_2, Y_2, \dots]$ be the restriction of $\rho_{\mathcal{B}}$.

Then $\rho^0_{\mathcal{B}}(\mathcal{W}^0)_0 = 1$ and for $r \geq 1$
$$\rho^0_{\mathcal{B}}(\mathcal{W}^0)_r \binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{\substack{k_1, \dots, k_r \geq 1 \\ m_1, \dots, m_r \geq 0}} b_{k_1} b_0^{m_1} \cdots b_{k_r} b_0^{m_r} X_1^{k_1-1} Y_1^{m_1} \cdots X_r^{k_r-1} Y_r^{m_r} .$$

$\boxed{\text{Theorem:}}$ For $*_b$ on $\mathbb{Q}\langle\mathcal{B}\rangle^0$ one obtains for $0 < n < r$ that

$$1 *_b \rho^0_{\mathcal{B}}(\mathcal{W}^0)_n = \rho^0_{\mathcal{B}}(\mathcal{W}^0)_n *_b 1 = \rho^0_{\mathcal{B}}(\mathcal{W}^0)_n ,$$

$$\rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_1, \dots, X_n}{Y_1, \dots, Y_n} *_b \rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_{n+1}, \dots, X_r}{Y_{n+1}, \dots, Y_r}$$
$$= \left(\rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_1, \dots, X_{n-1}}{Y_1, \dots, Y_{n-1}} *_b \rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_{n+1}, \dots, X_r}{Y_{n+1}, \dots, Y_r}\right) \cdot \rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_n}{Y_n + Y_r}$$
$$+ \left(\rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_1, \dots, X_n}{Y_1, \dots, Y_n} *_b \rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_{n+1}, \dots, X_{r-1}}{Y_{n+1}, \dots, Y_{r-1}}\right) \cdot \rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_r}{Y_n + Y_r}$$
$$+ \left(\rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_1, \dots, X_{n-1}}{Y_1, \dots, Y_{n-1}} *_b \rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_{n+1}, \dots, X_{r-1}}{Y_{n+1}, \dots, Y_{r-1}}\right)$$
$$\cdot \frac{\rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_n}{Y_n + Y_r} - \rho^0_{\mathcal{B}}(\mathcal{W}^0)\binom{X_r}{Y_n + Y_r}}{X_n - X_r}$$

Proof: As before, inductive and explicit calculations.

Step 2: Combining the maps ρ and φ

$\boxed{\text{Definition:}}$ Let $\rho_{\mathcal{A}}: \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}[Z_1, Z_2, \ldots]$ be a $\mathbb{Q}$-linear map as before, R be any $\mathbb{Q}$-algebra, and $\varphi: \mathbb{Q}\langle\mathcal{A}\rangle \to R$ be a $\mathbb{Q}$-linear map.

The (commutative) generating series associated to $(\varphi, \rho_{\mathcal{A}})$ are

$$(\varphi \otimes \rho_{\mathcal{A}})(\mathcal{W})_r(Z_1, \ldots, Z_{\ell(r)}) = \sum_{w \in (\mathcal{A}^*)^{(r)}} \varphi(w)\rho_{\mathcal{A}}(w) \in R[[Z_1, \ldots, Z_{\ell(r)}]]$$

we will often omit the index

for $r \geq 0$.

A particular kind of these generating series are known as moulds or bimoulds.

$\boxed{\text{Definition:}}$ (Ecalle) Let R be a $\mathbb{Q}$-algebra.

A sequence

$$M = (M_r(X_1, \ldots, X_r))_{r \geq 0} \in \prod_{r \geq 0} R[[X_1, \ldots, X_r]]$$

is called a mould with coefficients in R.

Similarly, a sequence

$$M = \left(M_r\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r}\right)_{r \geq 0} \in \prod_{r \geq 0} R[[X_1, Y_1, \ldots, X_r, Y_r]]$$

is called a bimould with coefficients in R.

Observation:

If $\rho_{\mathcal{A}}: \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}[z_1, z_2, \dots]$ is as before with $\ell(r) = r$ (resp. $\ell(r) = 2r$), and $\varphi: \mathbb{Q}\langle\mathcal{A}\rangle \to R$ is arbitrary, then $((\varphi \otimes \rho_{\mathcal{A}})(\mathcal{W})_r)_{r \geq 0}$ is a mould (resp. bimould) with coefficients in R.

$\boxed{\text{Definition:}}$ Let R be a $\mathbb{Q}$-algebra and

$$\ell(0) < \ell(1) < \ell(2) < \dots$$

a sequence of non-negative integers.

A sequence $M = (M_r)_{r \geq 0} \in \prod_{r \geq 0} R[[z_1, \dots, z_{\ell(r)}]]$ is called $(\varphi_{*_\diamond}, \rho_{\mathcal{A}})$-symmetric if there is

- a $\mathbb{Q}$-algebra morphism $\varphi_{*_\diamond}: (\mathbb{Q}\langle\mathcal{A}\rangle, *_\diamond) \to R$,
- a $\mathbb{Q}$-linear map $\rho_{\mathcal{A}}: \mathbb{Q}\langle\mathcal{A}\rangle \to \mathbb{Q}[z_1, z_2, \dots]$ as before,

s.th.

$$M_r = (\varphi_{*_\diamond} \otimes \rho_{\mathcal{A}})(\mathcal{W}(\mathcal{A}))_r \quad , \quad r \geq 0.$$

Special cases:

A bimould $M = (M_r)_{r \geq 0} \in \prod_{r \geq 0} R[[X_1, Y_1, \dots, X_r, Y_r]]$ is

- symmetril if there is an algebra morphism $\varphi_*: (\mathbb{Q}\langle\mathcal{A}^{bi}\rangle, *) \to R$, s.th. M is $(\varphi_*, \rho_{\mathcal{A}^{bi}})$-symmetric.

(this property was also intensively studied by Ecalle and Schneps)

- b-symmetril if there is an algebra morphism $\varphi_{*_b}: (\mathbb{Q}\langle\mathcal{B}\rangle^0, *_b) \to R$, s.th. $\mathcal{M}$ is $(\varphi_{*_b}, \rho^0_{\mathcal{B}})$-symmetric.

The explicit, recursive product formulas on the generating series of words allow to make symmetril and b-symmetril explicit.

Example:

- symmetril in depth 2 is

$$\mathcal{M}\binom{X_1}{Y_1}\cdot\mathcal{M}\binom{X_2}{Y_2} = \mathcal{M}\binom{X_1, X_2}{Y_1, Y_2} + \mathcal{M}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathcal{M}\binom{X_1}{Y_1+Y_2} - \mathcal{M}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}$$

- b-symmetril in depth 2 is

$$\mathcal{M}\binom{X_1}{Y_1}\cdot\mathcal{M}\binom{X_2}{Y_2} = \mathcal{M}\binom{X_2, X_1}{Y_2, Y_1+Y_2} + \mathcal{M}\binom{X_1, X_2}{Y_1, Y_1+Y_2} + \frac{\mathcal{M}\binom{X_1}{Y_1+Y_2} - \mathcal{M}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}$$

$\boxed{\text{Reminder / Theorem:}}$ The bimould of the combinatorial bi-multiple Eisenstein series $\mathcal{G} = (\mathcal{G}_r)_{r\geq 0}$ given by

$$\mathcal{G}_r\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} = \sum_{\substack{k_1, \ldots, k_r \geq 1 \\ d_1, \ldots, d_r \geq 0}} G\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} X_1^{k_1-1}\frac{Y_1^{d_1}}{d_1!} \cdots X_r^{k_r-1}\frac{Y_r^{d_r}}{d_r!}$$

(and $\mathcal{G}_0 = 1$) is symmetril.

This is equivalent to

$$(\mathbb{Q}\langle\mathcal{A}^{bi}\rangle, *) \longrightarrow \mathbb{Q}[[q]], \quad \begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix} \longmapsto G\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r}$$

being an algebra morphism. (see Part 1)

Step 1': The maps φ

Alternatively, we can also apply first the evaluation map φ to the generic diagonal series.

$\boxed{\text{Definition:}}$ Let R be a $\mathbb{Q}$-algebra and $\varphi: \mathbb{Q}\langle\mathcal{A}\rangle \to R$ be a $\mathbb{Q}$-linear map. The (non-commutative) generating series with coefficients in R associated to φ is

$$\varphi(\mathcal{W}(\mathcal{A})) = \sum_{w \in \mathcal{A}^*} \varphi(w)\, w \in R\langle\langle \mathcal{A} \rangle\rangle$$

completion of $R \otimes \mathbb{Q}\langle\mathcal{A}\rangle$

Particular kinds of these non-commutative generating series are the main objects in part 3. Essential will be the following observation.

$\boxed{\text{Proposition:}}$ Assume that $(\mathbb{Q}\langle\mathcal{A}\rangle, *_\diamond)$ is graded with $\deg(a) \geq 1$ $\forall a \in \mathcal{A}$.
Then φ is an algebra morphism for $*_\diamond$ iff $\varphi(\mathcal{W}(\mathcal{A}))$ is grouplike for the dual coproduct $\Delta_{*_\diamond}$ to $*_\diamond$.

2.3. Comparing the different descriptions of $\mathfrak{g}^f$

Having the general setup to treat generating series for quasi-shuffle algebras, we want to give now the isomorphism between the definition of $\mathfrak{g}^f$ in Part 1 and the definition in terms of the balanced setup.

$\boxed{\text{Definition:}}$ For any bimould $M=(M_r)_{r\geq 0}$, we define the bimould $M^{\#_y}=(M_r^{\#_y})_{r\geq 0}$ by

$$M_r^{\#_y}\begin{pmatrix}x_1,\dots,x_r\\ y_1,\dots,y_r\end{pmatrix} = M_r\begin{pmatrix}x_1, & x_2, & \dots, & x_r\\ y_1, & y_1+y_2, & \dots, & y_1+\dots+y_r\end{pmatrix}$$

(Evidently, the inverse operation is

$$M_r^{\#_y^{-1}}\begin{pmatrix}x_1,\dots,x_r\\ y_1,\dots,y_r\end{pmatrix} = M_r\begin{pmatrix}x_1, & x_2, & \dots, & x_r\\ y_1, & y_2-y_1, & \dots, & y_r-y_{r-1}\end{pmatrix}\Bigg)$$

$\boxed{\text{Definition:}}$ Let $\varphi_{\#}:\mathbb{Q}\langle\mathcal{A}^{bi}\rangle\to\mathbb{Q}\langle\mathcal{B}\rangle^{\circ}$ be the $\mathbb{Q}$-linear map implicitly defined by

$$(\varphi_{\#}\otimes\rho_{\mathcal{A}^{bi}})(\mathcal{W}(\mathcal{A}^{bi}))=\rho^{\circ}_{\mathcal{B}}(\mathcal{W}(\mathcal{B})^{\circ})^{\#_y}.$$

This means, the coefficient of $X_1^{k_1-1}\frac{Y_1^{d_1}}{d_1!}\cdots X_r^{k_r-1}\frac{Y_r^{d_r}}{d_r!}$ in $\rho^{\circ}_{\mathcal{B}}(\mathcal{W}^{\circ})_r^{\#_y}$ equals $\varphi_{\#}\left(\begin{bmatrix}k_1,\dots,k_r\\ d_1,\dots,d_r\end{bmatrix}\right)$.

Example:

$$\varphi_{\circledast}\left(\left[\begin{smallmatrix}k\\ d\end{smallmatrix}\right]\right) = d!\, b_k b_0^d$$

$$\varphi_{\circledast}\left(\left[\begin{smallmatrix}k_1,k_2\\ d_1,d_2\end{smallmatrix}\right]\right) = \sum_{n=d_2}^{d_1+d_2} \frac{d_1!\,n!}{(n-d_2)!}\, b_{k_1} b_0^{d_1+d_2-n} b_{k_2} b_0^{n}$$

$\boxed{\text{Theorem:}}$ (B.) The map

$$\varphi_{\circledast} : (\mathbb{Q}\langle\mathcal{A}^{bi}\rangle, \ast) \longrightarrow (\mathbb{Q}\langle\mathcal{B}\rangle^{0}, \ast_b)$$

is an isomorphism of weight-graded algebras satisfying

$$\varphi_{\circledast} \circ \sigma = \tau \circ \varphi_{\circledast}$$

Idea of proof: Use generating series

- Use the explicit formulas for $\ast$ and $\ast_b$ on generating series of words to obtain that $\varphi_{\circledast}$ is an algebra morphism.
- $\varphi_{\circledast} \circ \sigma = \tau \circ \varphi_{\circledast}$ can be verified quite easily on generating series $(\square)$

$\boxed{\text{Corollary:}}$ We have

$$(\mathbb{Q}\langle\mathcal{A}^{bi}\rangle, \ast)\big/\langle w-\sigma(w) \mid w\in\mathbb{Q}\langle\mathcal{A}^{bi}\rangle\rangle \cong (\mathbb{Q}\langle\mathcal{B}\rangle^{0}, \ast_b)\big/\langle \tau(w)-w \mid w\in\mathbb{Q}\langle\mathcal{B}\rangle^{0}\rangle$$

$(= \mathcal{G}^{f})$

$\boxed{\text{Corollary:}}$ A bimould M is b-symmetril and τ-invariant if and only if $M^{\#_Y}$ is symmetril and swap invariant.

Applying $\#_Y$ to the bimould of the combinatorial bi-multiple Eisenstein series, one obtains the bimould of the balanced multiple q-zeta values. Hence, those are τ-invariant and satisfy the balanced quasi-shuffle product formula.