Finite multiple zeta values

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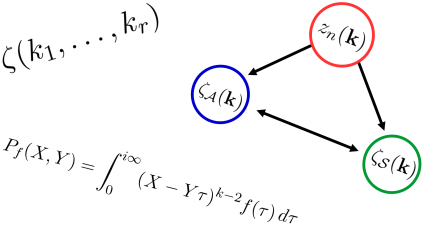

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Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ we define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight k.

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}\,.$$

There are two different ways to express the product of MZV in terms of MZV.

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j, k_1+k_2-j)\,.$$

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)$$
$$\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0}\frac{1}{m^2 n} = \zeta(2,1) = \zeta(3) = \sum_{m>0}\frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations

$\rightsquigarrow$**extended double shuffle relations**.

MZV Conjectures

- The extended double shuffle relations give all linear relations among MZV and

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k \,,$$

  i.e. there are no relations between MZV of different weight.

- (Zagier) The dimension of the spaces $\mathcal{Z}_k$ is given by

$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k \, X^k = \frac{1}{1 - X^2 - X^3} \,.$$

- (Hoffman) The following set gives a basis of $\mathcal{Z}$

$$\{\zeta(k_1, \dots, k_r) \mid r \geq 0, k_1, \dots, k_r \in \{2,3\}\} \,.$$

Theorem (Deligne-Goncharov, Terasoma)

We have $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$, where $\sum_{k\geq 0} d_k\, X^k := (1 - X^2 - X^3)^{-1}$.

Theorem (Brown, 2012)

Every MZV can be written as a linear combination of $\zeta(k_1, \ldots, k_r)$ with $k_j \in \{2, 3\}$.

Example

$$\zeta(4) = \frac{4}{3}\zeta(2,2)\,, \qquad \zeta(5) = \frac{6}{5}\zeta(2,3) + \frac{4}{5}\zeta(3,2)\,,$$

$$\zeta(4,1) = \frac{1}{5}\zeta(2,3) - \frac{1}{5}\zeta(3,2)\,, \qquad \zeta(6) = \frac{16}{3}\zeta(2,2,2)\,.$$

Theorem (Gangl-Kaneko-Zagier, 2006)

Modular forms of weight k "give" relations between $\zeta(r,s)$ and $\zeta(k)$ with $k=r+s$ and r,s odd.

There are explicit formulas for these relation using period polynomials (next slide).

Example

- Each Eisenstein series in weight k corresponds to the relation

$$\zeta(3,k-3)+\zeta(5,k-5)+\cdots+\zeta(k-3,3)+\zeta(k-1,1)=\frac{1}{4}\zeta(k)\,.$$

- The cusp form Δ in weight 12 gives

$$168\zeta(5,7)+150\zeta(7,5)+28\zeta(9,3)=\frac{5197}{691}\zeta(12)\,.$$

V_k: homogeneous polynomials of degree $k-2$.

Definition

For a cusp for $f \in S_k$ its **period polynomial** is defined as the following polynomial in $\mathbb{C} \otimes V_k$

$$P_f(X,Y) = \int_0^{i\infty} (X - Y\tau)^{k-2} f(\tau)\, d\tau\,.$$

Denote by P_f^- the even part of P_f. These are elements in $\mathbb{C} \otimes W_k^-$, where

$$W_k^- = \{P \in V_k \mid P(X,Y) - P(X+Y,Y) + P(X+Y,X) = 0\}\,.$$

Theorem (Eichler-Shimura-Manin)

The map $p^- : f \mapsto P_f^-$ induces an isomorphism

$$p^- : S_k \xrightarrow{\sim} \mathbb{C} \otimes {W_k^-} \big/ {\mathbb{Q}(X^{k-2} - Y^{k-2})}\,.$$

Define for a polynomial $p \in V_k$ the coefficients $\beta^p_{r,s} \in \mathbb{Q}$ by

$$\sum_{\substack{r+s=k\\ r,s\geq 1}} \binom{k-2}{r-1} \beta^p_{r,s} X^{r-1} Y^{s-1} := p(X+Y, Y)\,.$$

Then the more precise statement of the Theorem of Gangl-Kaneko-Zagier is as follows:

Theorem (Gangl-Kaneko-Zagier, 2006)

For all $p \in W_k^-$ with $k \geq 4$ even we have

$$\sum_{\substack{r+s=k\\ r,s\geq 1 \text{ odd}}} \beta^p_{r,s} \zeta(r,s) \equiv 0 \mod \mathbb{Q}\zeta(k)\,.$$

Definition

For $k_1, \dots, k_r \geq 1$ there exists a unique $\zeta(k_1, \dots, k_r; T) \in \mathcal{Z}[T]$ with

- $\zeta(1; T) = T$,
- For $k_1 \geq 2$ it is $\zeta(k_1, \dots, k_r; T) = \zeta(k_1, \dots, k_r)$,
- Their product can be expressed by the harmonic product formula.

Example

Since

$$\zeta(1; T) \cdot \zeta(2; T) = \zeta(1, 2; T) + \zeta(2, 1; T) + \zeta(3; T)$$

we have

$$\zeta(1, 2; T) = \zeta(2)T - \zeta(2, 1) - \zeta(3)\,.$$

In general we have for $\mathbf{k}$ admissible: $\zeta(\underbrace{1, \dots, 1}_{m}, \mathbf{k}; T) = \zeta(\mathbf{k})\frac{T^m}{m!} + \dots$.

Definition

For an indexset $\mathbf{k} = (k_1, \ldots, k_r)$ define the **symmetric multiple zeta value** by

$$\zeta_{\mathcal{S}}(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta(k_a, k_{a-1}, \ldots, k_1; T)\zeta(k_{a+1}, k_{a+2}, \ldots, k_r; T)\,.$$

- One can check that the definition of $\zeta_{\mathcal{S}}$ is independent of T.
- The product of two SMZV can again be expressed by the harmonic product, e.g.

$$\zeta_{\mathcal{S}}(k_1) \cdot \zeta_{\mathcal{S}}(k_2) = \zeta_{\mathcal{S}}(k_1, k_2) + \zeta_{\mathcal{S}}(k_2, k_1) + \zeta_{\mathcal{S}}(k_1 + k_2)\,.$$

In depth $r = 1$ we have for $k \geq 1$

$$\zeta_{\mathcal{S}}(k) = \zeta(k;T) + (-1)^k \zeta(k;T) = \begin{cases} 2\zeta(k) & , k \text{ is even} \\ 0 & , k \text{ is odd} \end{cases} .$$

Question: Do we get all MZV?

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Question: Do we get all MZV?

Theorem (Yasuda, 2014)

We have $\mathcal{Z} = \langle \zeta_{\mathcal{S}}(\mathbf{k}) \rangle_{\mathbb{Q}}$.

Relations between MZV give relation between Symmetric MZV:

Example

$$\zeta(5) - 2\zeta(2,3) + 4\zeta(4,1) = 0$$
$$\iff$$
$$\zeta_{\mathcal{S}}(4,1) - \zeta_{\mathcal{S}}(1,4) + \zeta_{\mathcal{S}}(3,2) = 0$$

$$\sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$

Definition

For an indexset $\mathbf{k} = (k_1, \ldots, k_r)$ the **finite multiple zeta value** is defined by

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \left(\sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_{p \text{ prime}} \in \mathcal{A},$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = \prod_{p \text{ prime}} \mathbb{F}_p \Big/ \bigoplus_{p \text{ prime}} \mathbb{F}_p .$$

$(\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z})$

② Finite MZV - The algebra $\mathcal{A}$

We have an embedding $\mathbb{Q} \overset{i}{\hookrightarrow} \mathcal{A}$, since for $\frac{a}{b} \in \mathbb{Q}$ we can get a solution x_p of

$$b\,x_p - a \equiv 0 \mod p$$

for all but finitely many p. Set $x_p = 0$ if it does not exists and define

$$i\left(\frac{a}{b}\right) = (x_2, x_3, x_5, x_7, \dots) \in \mathcal{A} = \prod_{p \text{ prime}} \mathbb{F}_p \Big/ \bigoplus_{p \text{ prime}} \mathbb{F}_p\,.$$

$\implies \mathcal{A}$ **is a $\mathbb{Q}$-algebra.**

Example

$$i\left(\frac{3}{10}\right) = (0, 0, 0, 1, 8, 12, 2, 6, 21, \dots)\,.$$

For the space spanned by all FMZVs we write

$$\mathcal{Z}^{\mathcal{A}} = \langle \zeta_{\mathcal{A}}(\mathbf{k}) \rangle_{\mathbb{Q}} \,.$$

Finite MZV satisfy the same harmonic product formula as MZV, e.g.

$$\zeta_{\mathcal{A}}(k_1) \cdot \zeta_{\mathcal{A}}(k_2) = \zeta_{\mathcal{A}}(k_1, k_2) + \zeta_{\mathcal{A}}(k_2, k_1) + \zeta_{\mathcal{A}}(k_1 + k_2)$$

and therefore $\mathcal{Z}^{\mathcal{A}}$ is a $\mathbb{Q}$-algebra.

Proposition

- Depth 1: For $k \geq 1$ we have $\zeta_{\mathcal{A}}(k) = 0$.
- Depth 2: For $k_1, k_2 \geq 1$ we have

$$\zeta_{\mathcal{A}}(k_1, k_2) = \left((-1)^{k_1} \binom{k_1 + k_2}{k_2} \frac{B_{p-k_1-k_2}}{k_1 + k_2} \right)_{p \text{ prime}} .$$

- Clearly $\zeta_{\mathcal{A}}(k_1, k_2) = 0$ if $k_1 + k_2$ is even.
- It is expected, that $\zeta_{\mathcal{A}}(k_1, k_2) \neq 0$ if $k_1 + k_2$ is odd.
- We do not know an example for $\mathbf{k} \neq \emptyset$, for which we can prove $\zeta_{\mathcal{A}}(\mathbf{k}) \neq 0$.

In their work, Kaneko and Zagier prove several linear relations among finite MZV.

Example

$$\zeta_{\mathcal{A}}(4,1) - \zeta_{\mathcal{A}}(1,4) + \zeta_{\mathcal{A}}(3,2) = 0$$

They also made the following observation

Observation (Kaneko, Zagier)

The number of relations between $\zeta_{\mathcal{A}}(2a,1,2b,1)$ seems to correspond to cusp forms in weight $2(a+b+1)$.

For example in weight 12 the first relation of this type is given by

$$16\zeta_{\mathcal{A}}(2,1,8,1) + 9\zeta_{\mathcal{A}}(4,1,6,1) + 18\zeta_{\mathcal{A}}(6,1,4,1) - 2\zeta_{\mathcal{A}}(8,1,2,1) = 0\,.$$

There are no proven results on this observation or on any connections of finite MZV with modular forms.

Conjecture (Kaneko-Zagier)

- We have an $\mathbb{Q}$-algebra isomorphism

$$\begin{aligned} \varphi_{KZ} : \mathcal{Z}^{\mathcal{A}} &\longrightarrow \mathcal{Z}/\pi^2\mathcal{Z} \\ \zeta_{\mathcal{A}}(\mathbf{k}) &\longmapsto \zeta_{\mathcal{S}}(\mathbf{k}) \mod \pi^2\mathcal{Z}\,. \end{aligned}$$

- The dimension of $\mathcal{Z}_k^{\mathcal{A}}$ is given by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k^{\mathcal{A}} X^k = \frac{1-X^2}{1-X^2-X^3}\,.$$

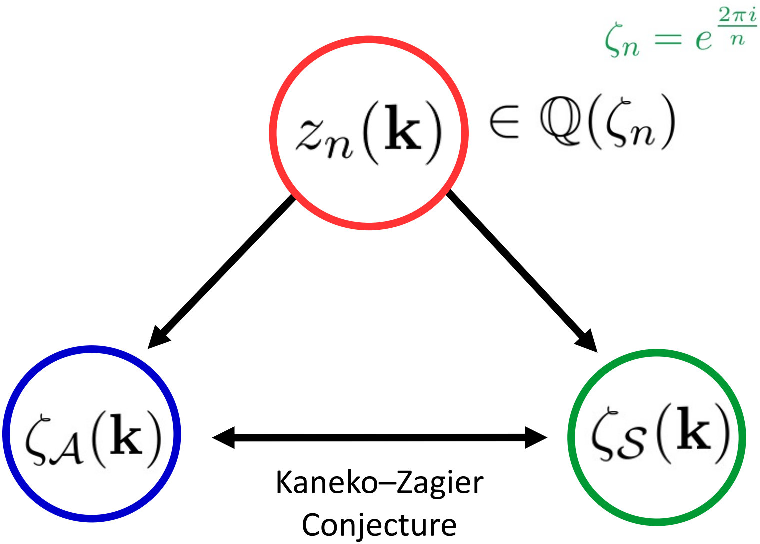
- We do not even know if the map φ_{KZ} is well-defined.

$\zeta_{\mathcal{A}}(\mathbf{k})$ ⟷ $\zeta_{\mathcal{S}}(\mathbf{k})$

Kaneko–Zagier
Conjecture

$(a_2, a_3, a_5, \dots)$
$a_p \in \mathbb{F}_p$

$\in \mathbb{R}$


$\zeta_n = e^{\frac{2\pi i}{n}}$
$z_n(\mathbf{k}) \in \mathbb{Q}(\zeta_n)$
$\zeta_{\mathcal{A}}(\mathbf{k})$
$\zeta_{\mathcal{S}}(\mathbf{k})$
Kaneko–Zagier
Conjecture

Definition

For $n \geq 1$ and an index set $\mathbf{k} = (k_1, \ldots, k_r)$ with $k_1, \ldots, k_r \geq 1$ we define

$$z_n(\mathbf{k}; q) = z_n(k_1, \ldots, k_r; q) = \sum_{n > m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1} \ldots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}},$$

where $[m]_q = \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1}$.

1. Notice that for $k_1 \geq 2$

$$\lim_{n\to\infty} \lim_{q\to 1} z_n(\mathbf{k}; q) = \zeta(\mathbf{k}) .$$

2. We will be interested in the values $z_n(\mathbf{k}; \zeta_n) \in \mathbb{Q}(\zeta_n)$, where ζ_n is a primitive n-th root of unity.

③ Multiple harmonic q-series - "Analytic limit" ($n \to \infty$)

Theorem (B.-Takeyama-Tasaka, 2018)

For any index set $\mathbf{k} = (k_1, \ldots, k_r)$ the limit $\lim_{n\to\infty} z_n(\mathbf{k}; e^{\frac{2\pi i}{n}})$ exists and we set

$$\xi(\mathbf{k}) := \lim_{n\to\infty} z_n(\mathbf{k}; e^{\frac{2\pi i}{n}}) \in \mathbb{C}\,.$$

It is given by

$$\xi(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta\big(k_a, k_{a-1}, \ldots, k_1; \frac{\pi i}{2}\big) \zeta\big(k_{a+1}, k_{a+2}, \ldots, k_r; -\frac{\pi i}{2}\big)\,.$$

Corollary

For any index set $\mathbf{k} = (k_1, \ldots, k_r)$ we have

$$\mathrm{Re}\,(\xi(\mathbf{k})) \equiv \zeta_{\mathcal{S}}(\mathbf{k}) \mod \pi^2 \mathcal{Z}\,.$$

$\rightsquigarrow$ Relations among the z_n give relations among $\zeta_{\mathcal{S}}$ (modulo $\pi^2 \mathcal{Z}$).

Theorem (B.-Takeyama-Tasaka, 2018)

For any primitive root of unity ζ_p, we have

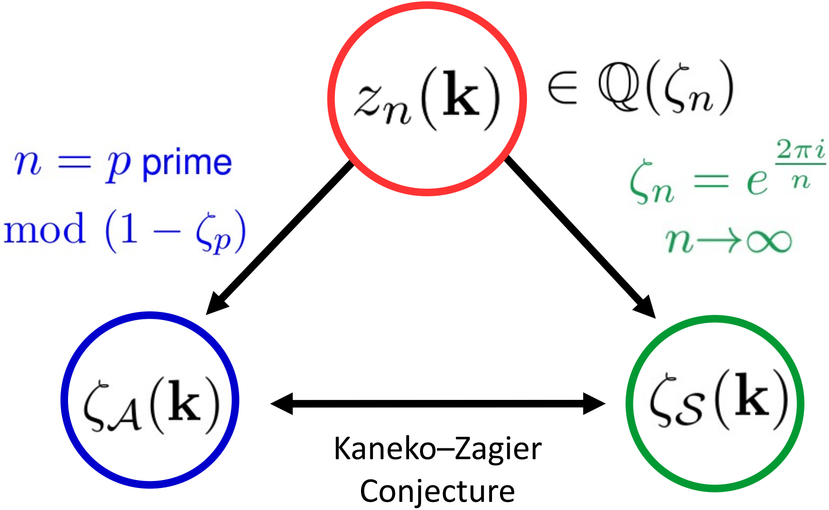
$$(z_p(\mathbf{k}) \mod \mathfrak{p})_p = \zeta_{\mathcal{A}}(\mathbf{k}),$$

where $\mathfrak{p} = (1-\zeta_p)$ is the prime ideal of $\mathbb{Z}[\zeta_p]$ generated by $1-\zeta_p$.

Proof:

- For p prime it is $z_p(\mathbf{k}) \in \mathbb{Z}[\zeta_p]$.
- It holds that $\mathbb{Z}[\zeta_p]/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$.
- For $p > m > 0$ we have $[m]_{\zeta_p} \equiv m \mod \mathfrak{p}$.

$\rightsquigarrow$ Relations among the z_p give relations among $\zeta_{\mathcal{A}}$.


$z_n(\mathbf{k})$
$\in \mathbb{Q}(\zeta_n)$
$n = p$ prime
mod $(1 - \zeta_p)$
$\zeta_n = e^{\frac{2\pi i}{n}}$
$n \to \infty$
$\zeta_{\mathcal{A}}(\mathbf{k})$
$\zeta_{\mathcal{S}}(\mathbf{k})$
Kaneko–Zagier
Conjecture

1. There exist "level N" versions of (finite) multiple zeta values.
2. These are defined by introducing powers of N-th roots of unity in the numerator.

For example (some of) the level $N=2$ versions of finite double zeta values are defined as follows:

Definition

For $r, s \geq 1$ we define the following **alternating finite double zeta values**

$$\zeta_{\mathcal{A}}(\overline{r}, s) = \left(\sum_{p>m>n>0} \frac{(-1)^m}{m^r n^s} \mod p \right)_p \in \mathcal{A}\,,$$

$$\zeta_{\mathcal{A}}(\overline{r}, \overline{s}) = \left(\sum_{p>m>n>0} \frac{(-1)^{m+n}}{m^r n^s} \mod p \right)_p \in \mathcal{A}\,.$$

In a similar way one can define $\zeta_{\mathcal{A}}(\overline{k})$ and $\zeta_{\mathcal{A}}(r, \overline{s})$.

Define the space of finite alternating double zeta values by

$$\mathcal{F}_k = \langle \zeta_{\mathcal{A}}(r,s) \mid r,s \in \mathbb{N} \cup \overline{\mathbb{N}}, |r| + |s| = k \rangle_{\mathbb{Q}} + \mathbb{Q}\zeta_{\mathcal{A}}(\overline{k}),$$

where $\overline{\mathbb{N}} = \{\overline{1}, \overline{2}, \dots\}$ and $|\overline{r}| = r$.

Proposition

For odd k we have $\mathcal{F}_k = \mathbb{Q}\zeta_{\mathcal{A}}(\overline{k})$.

For even weight k it seems that the space $\mathcal{F}_k$ is not understood yet.

As before we define for a hom. polynomial $p \in \mathbb{Q}[X,Y]$ of degree $k-2$ the coefficients $\beta^p_{r,s} \in \mathbb{Q}$ by

$$\sum_{\substack{r+s=k\\ r,s\geq 1}} \binom{k-2}{r-1} \beta^p_{r,s} X^{r-1} Y^{s-1} := p(X+Y,Y)\,.$$

Theorem (B.-Anzawa, 2021+)

For any even hom. polynomial $p \in \mathbb{Q}[X,Y]$ of degree $k-2$ we have

$$\sum_{\substack{r+s=k\\ r,s\geq 1}} \beta^p_{r,s} 2^r \zeta_{\mathcal{A}}(\overline{r}, s) = 0\,.$$

For example for $p(X,Y) = X^{k-2}$ with $k \geq 4$ even we obtain

$$\sum_{\substack{r+s=k\\ r\geq 2, s\geq 1}} 2^r \zeta_{\mathcal{A}}(\overline{r}, s) = 0\,.$$

④ Alternating finite MZV - Observations

1. In contrast to classical double zeta values the theorem gives relations for all even polynomials and not just period polynomials.
2. (It seems like that) this theorem gives not all relations in even weight. For example, numerical experiments suggest that

$$21\zeta_{\mathcal{A}}(\overline{4},2) \stackrel{?}{=} -8\zeta_{\mathcal{A}}(\overline{3},3) - 36\zeta_{\mathcal{A}}(\overline{5},1)\,.$$

3. In general we expect that for even k we have

$$\mathcal{F}_k \stackrel{?}{=} \langle \zeta_{\mathcal{A}}(\overline{r},s) \mid r,s \geq 1 \text{ odd}\,, r+s=k, \rangle_{\mathbb{Q}}\,.$$

Moreover it seems that all the $\zeta_{\mathcal{A}}(\overline{r},s)$ for r,s odd are linearly independent.

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Moreover it seems that all the $\zeta_{\mathcal{A}}(\overline{r},s)$ for r,s odd are linearly independent.

Thank you very much for your attention!