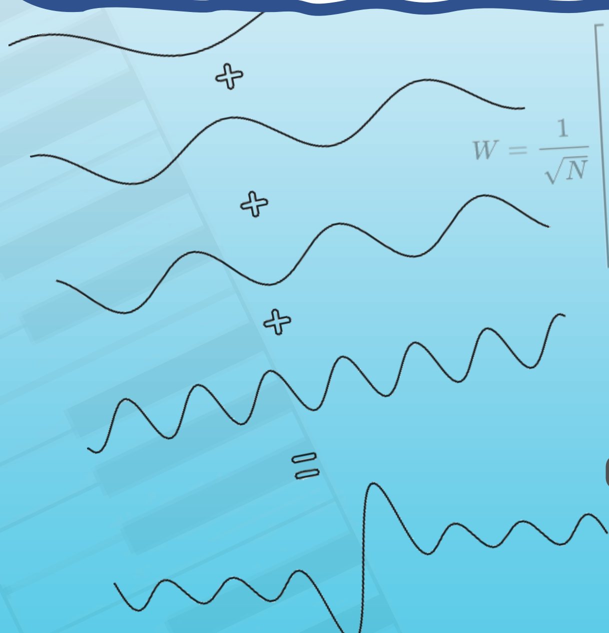
From Playstation to Hospitals

Hidden mathematics in our daily life

Studium Generale - 8th November 2019



$$W = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

$$Rf(\gamma) = \int_\gamma f(x,y)\,ds$$

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Graduate school of mathematics

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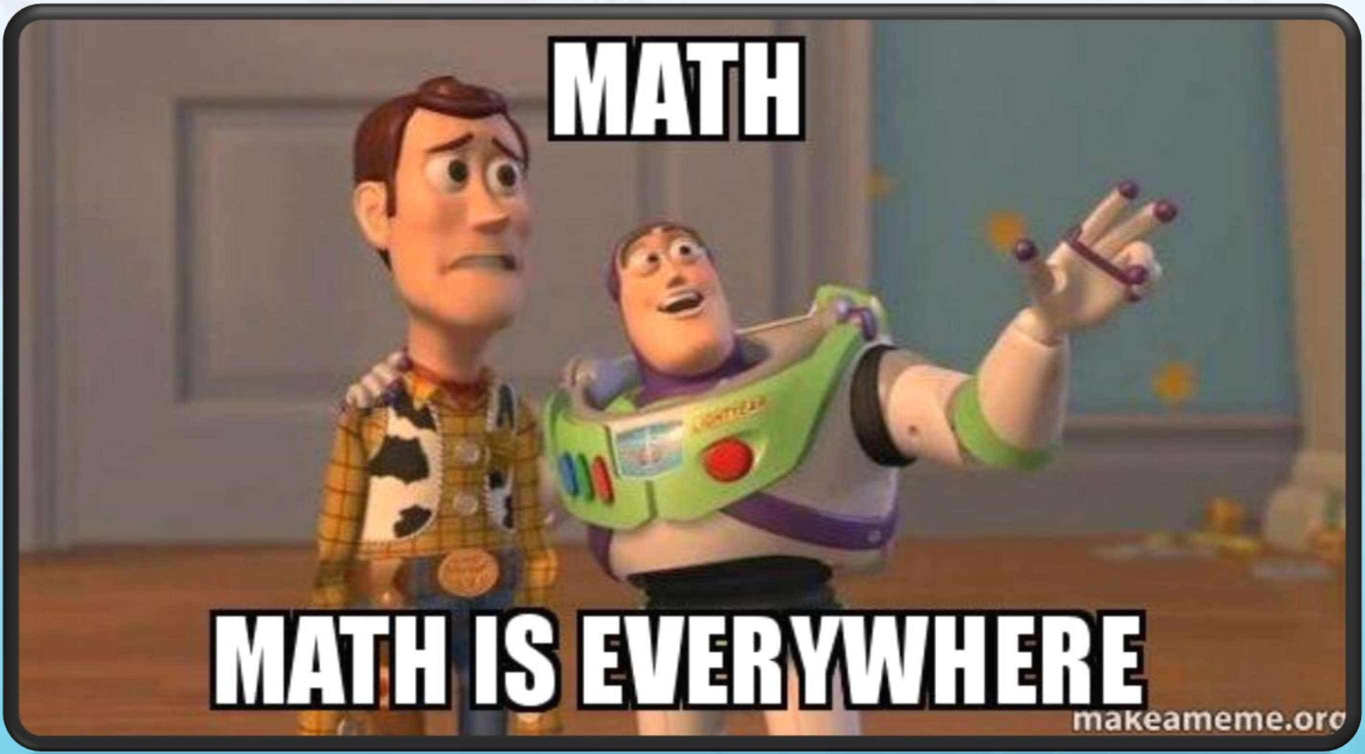
About me

- Born in Hamburg (Germany) 
- Studied mathematics at Hamburg University
- Since last month Associate Professor at Nagoya University in the G30 Program
- Interested in Number theory

Mathematics….



Do you like mathematics?

Yes

I always liked mathematics in school / university. It is a really cool subject!

No

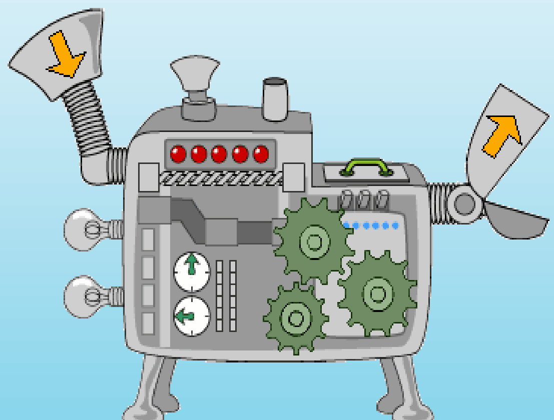
I accept that mathematics might be important, but I was never a big fan of it.

Ein Beispiel / An Example

Heute möchte ich über Mathematik sprechen

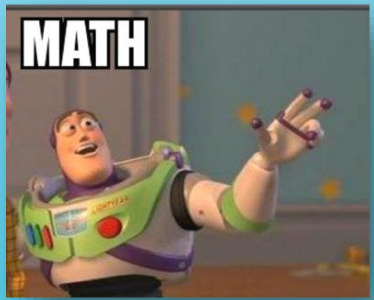
Today want I about math talk

„Heute möchte ich über Mathematik sprechen“



“Today I want to talk about math”

Google Translate



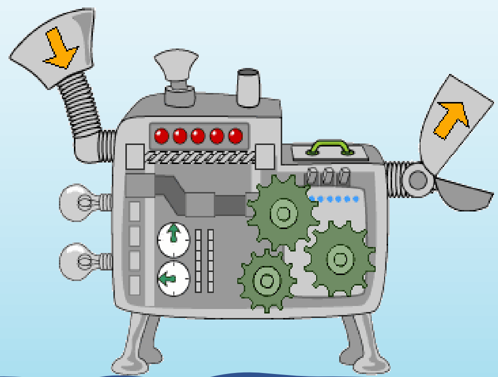
The magic machine

In general we often have something like this….

Input

Data we can not understand

e.g.
- Text in different language
- Something we measure by sensors
- ….



Magic machine

Output

Data we can understand (and manipulate)

Often the „inverse“ machine also exists

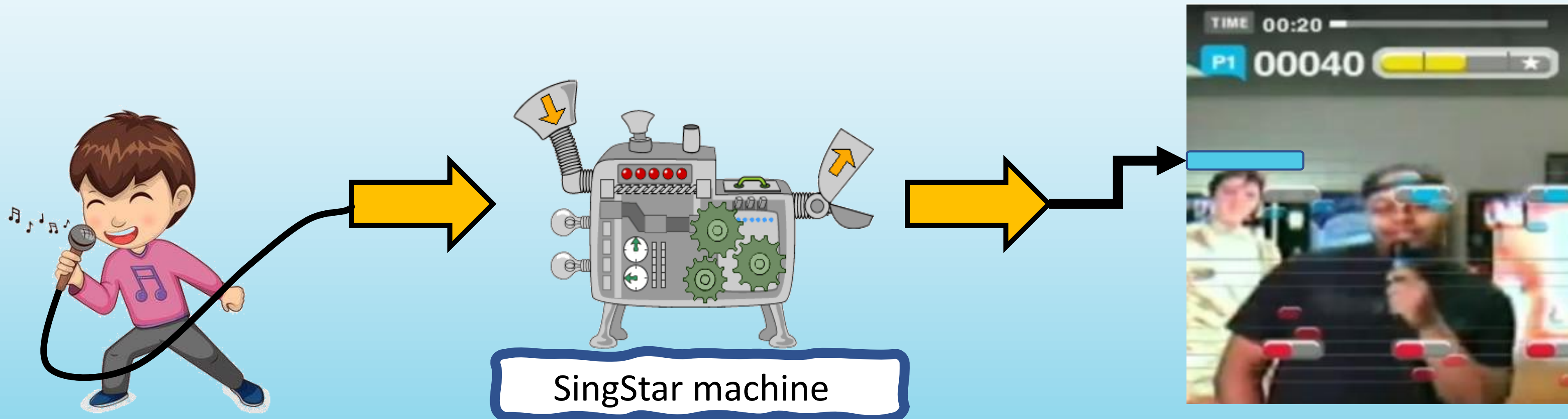
Today: Discuss two explicit examples

Karaoke / Playstation SingStar

- Playstation SingStar is a competitive karaoke game.
- You score by singing a song in the correct pitch.

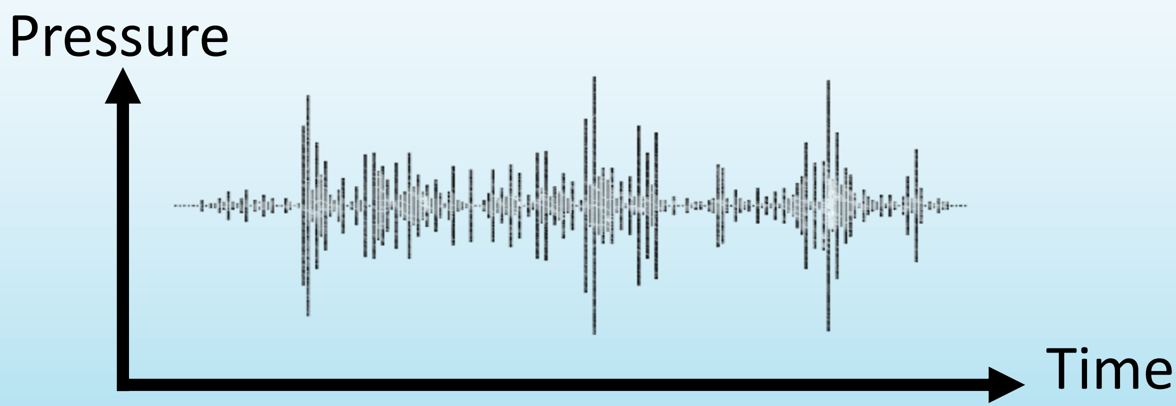
Also appears on Japanese TV

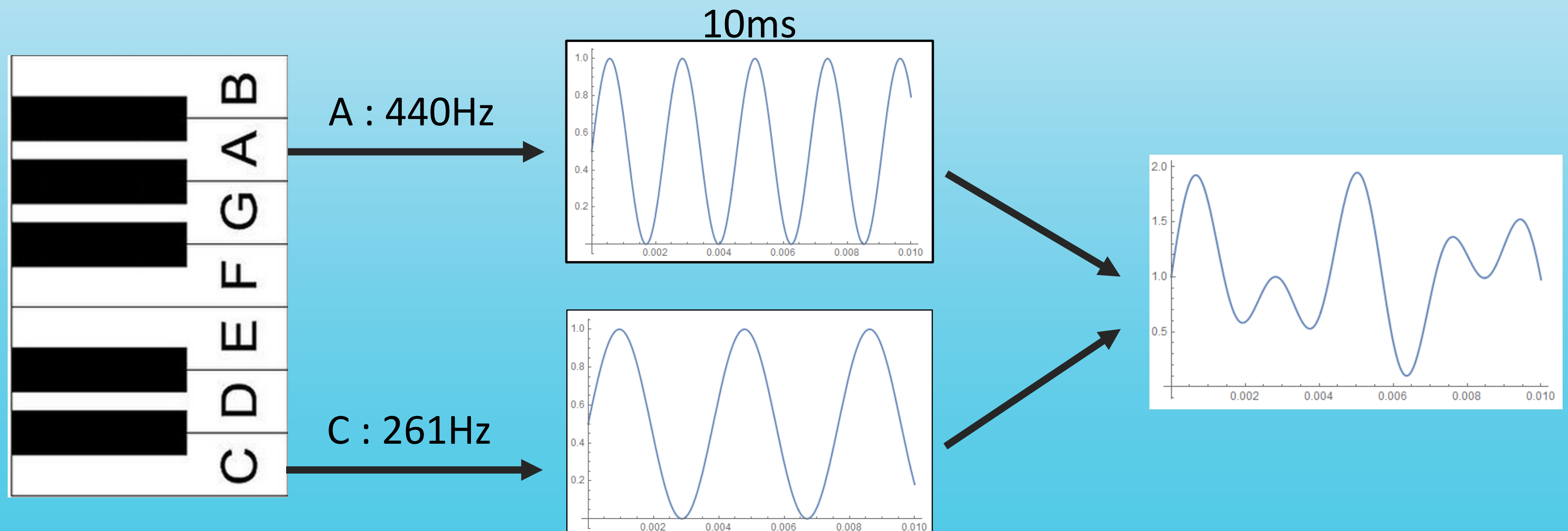
SingStar: How does it work?



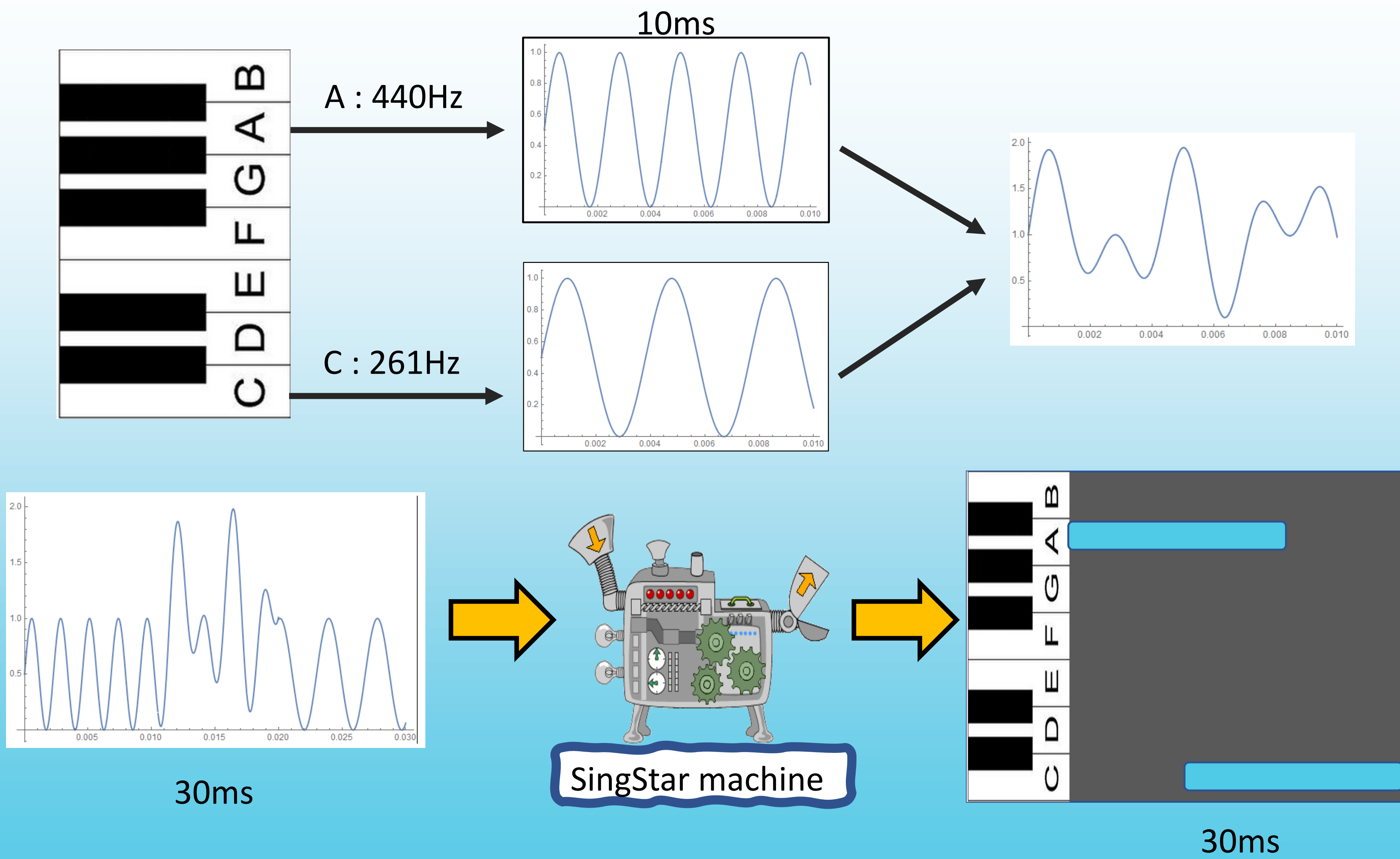
Soundwaves

Sound is a vibration that typically propagates as an audible wave of pressure, through a transmission medium such as a gas, liquid or solid.

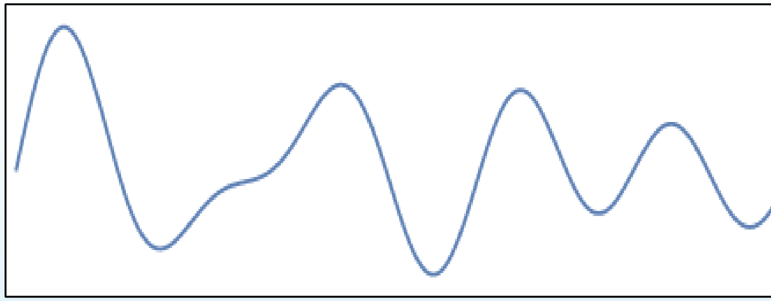




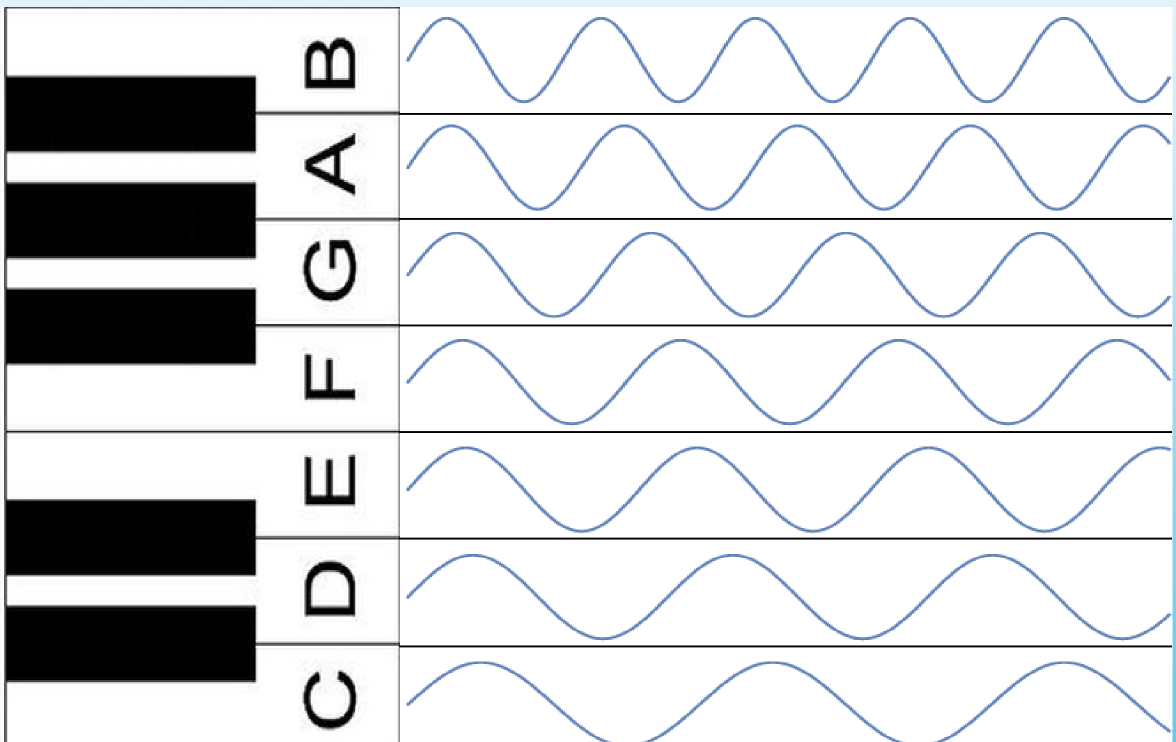
SingStar: Piano version



Can you be a SingStar machine?



Which three keys are played here?

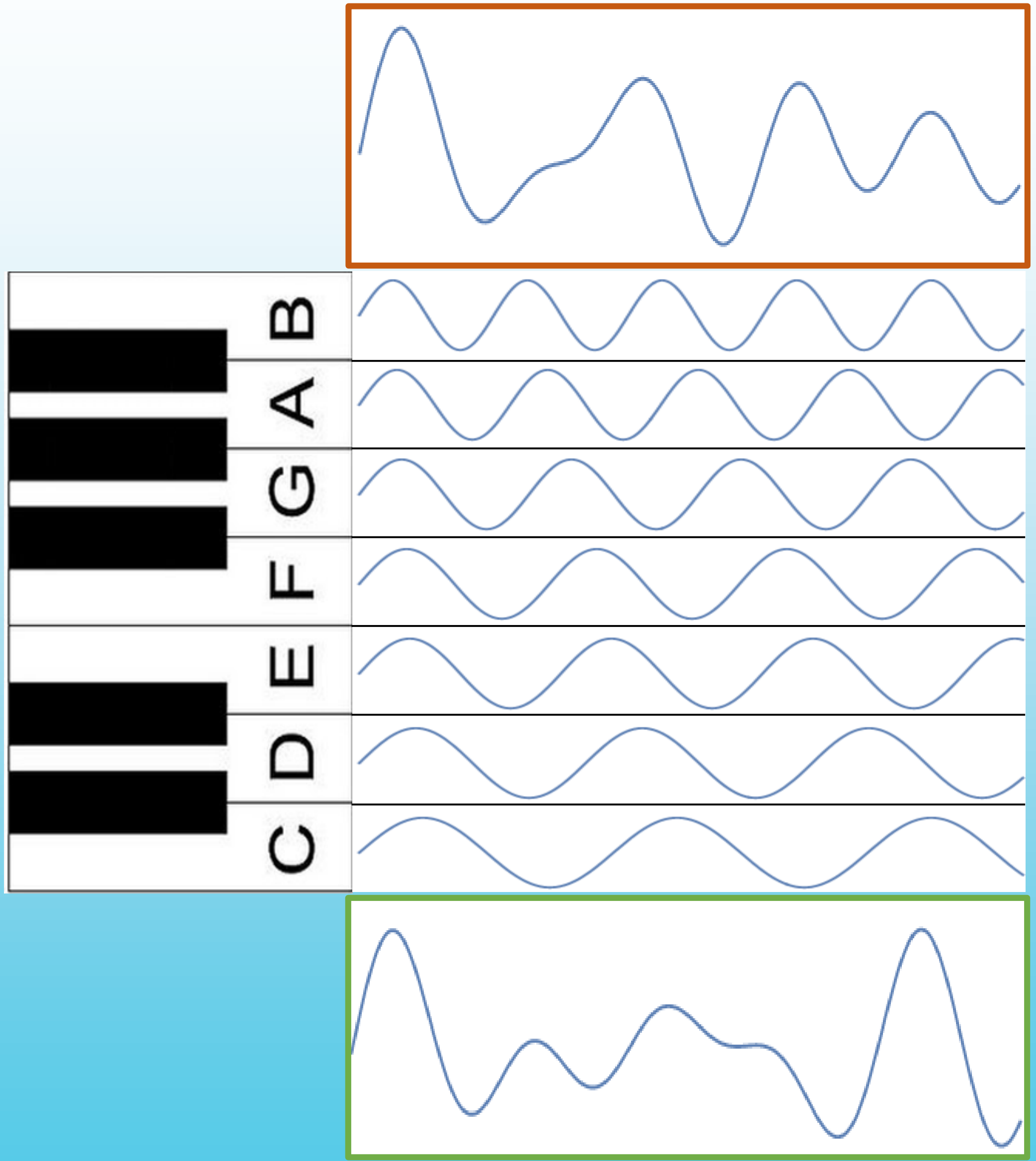


10ms

Yes: C + G + B

No: D + F + B

Can you be a SingStar machine?

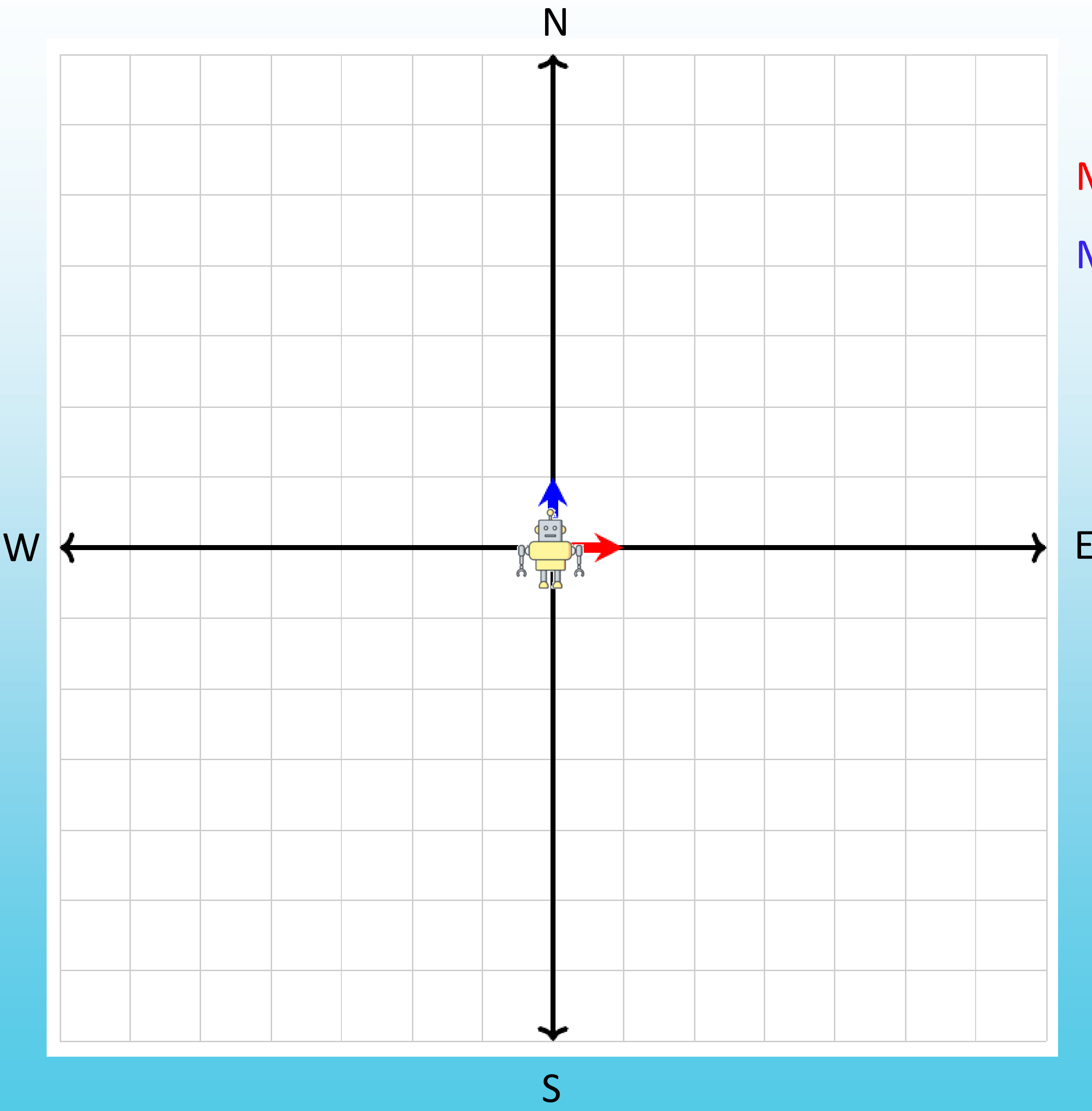


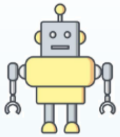
Correct: No

D + F + B

C + G + B

Some linear algebra...

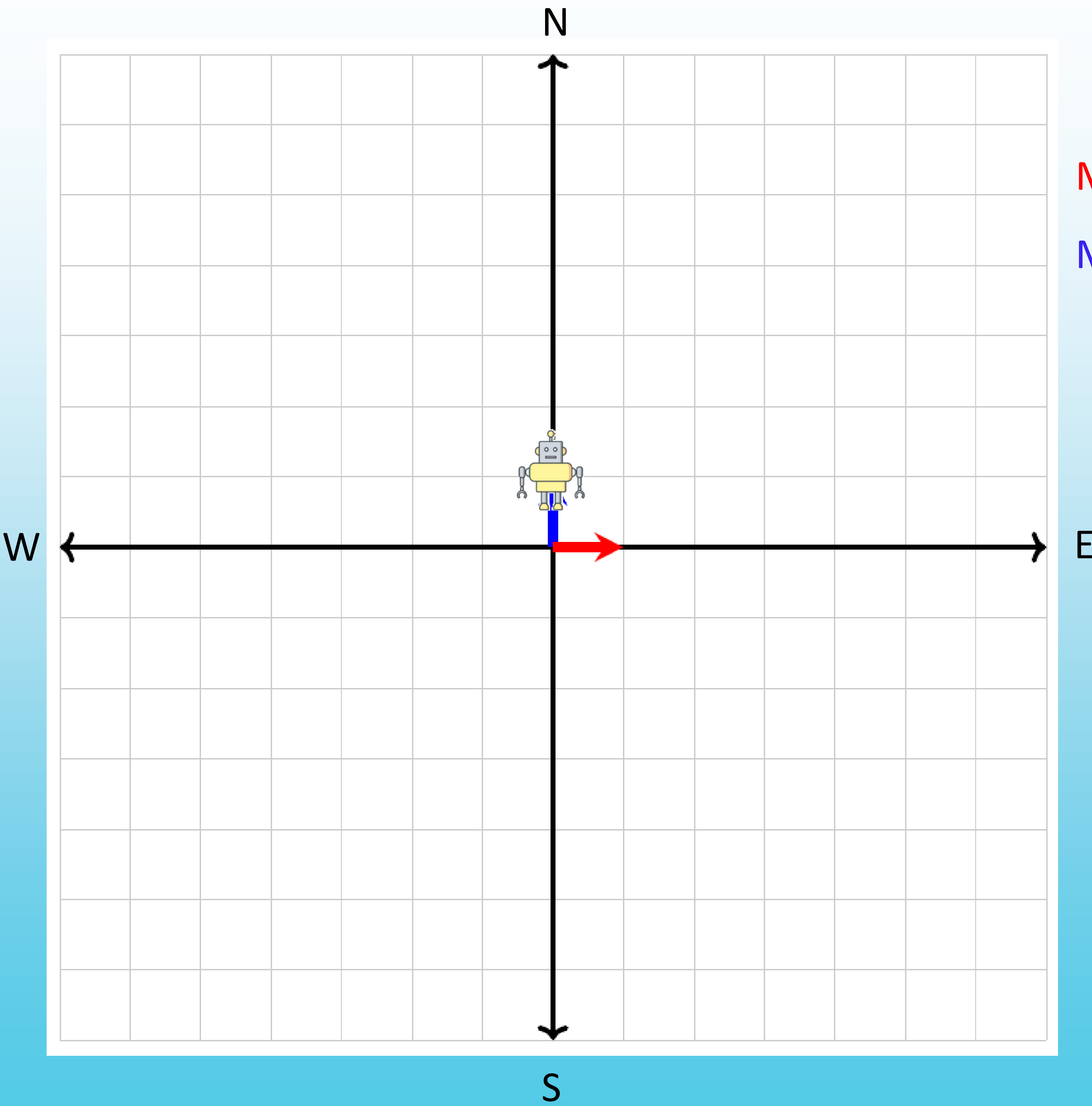


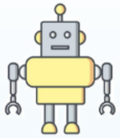


Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

Some linear algebra…

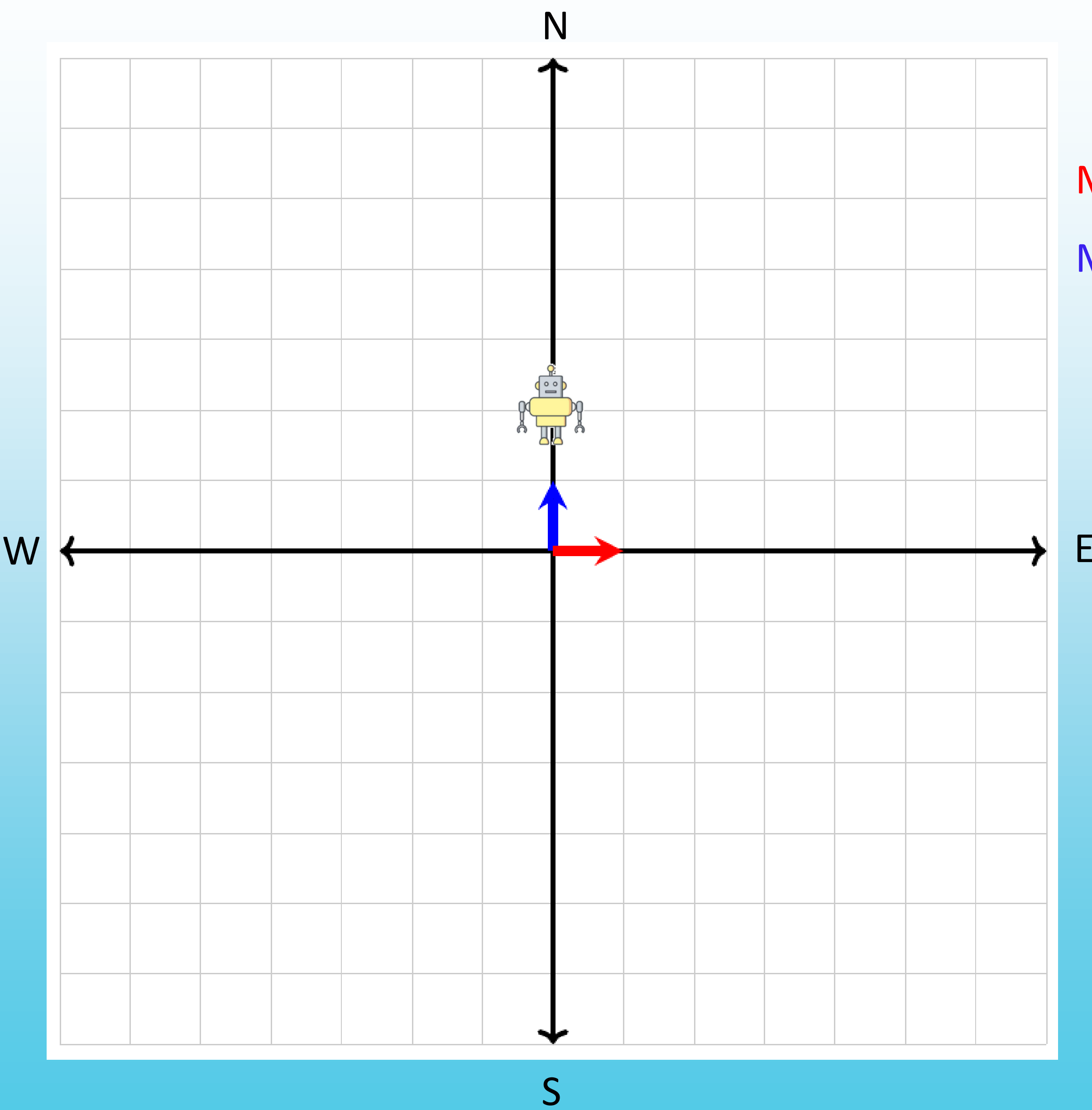


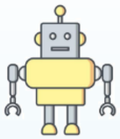


Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

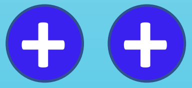
Some linear algebra...



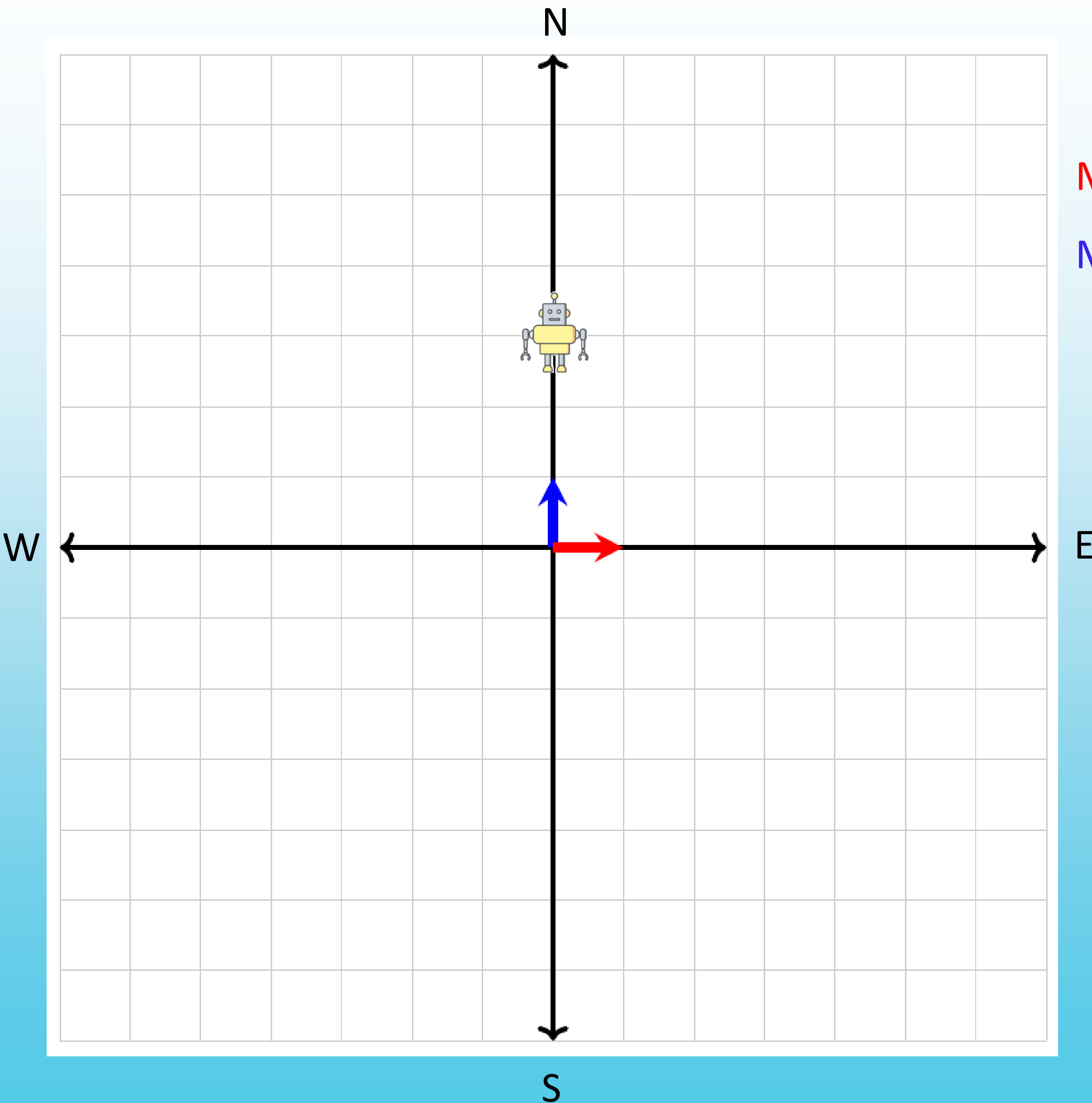


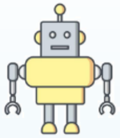
Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north



Some linear algebra…

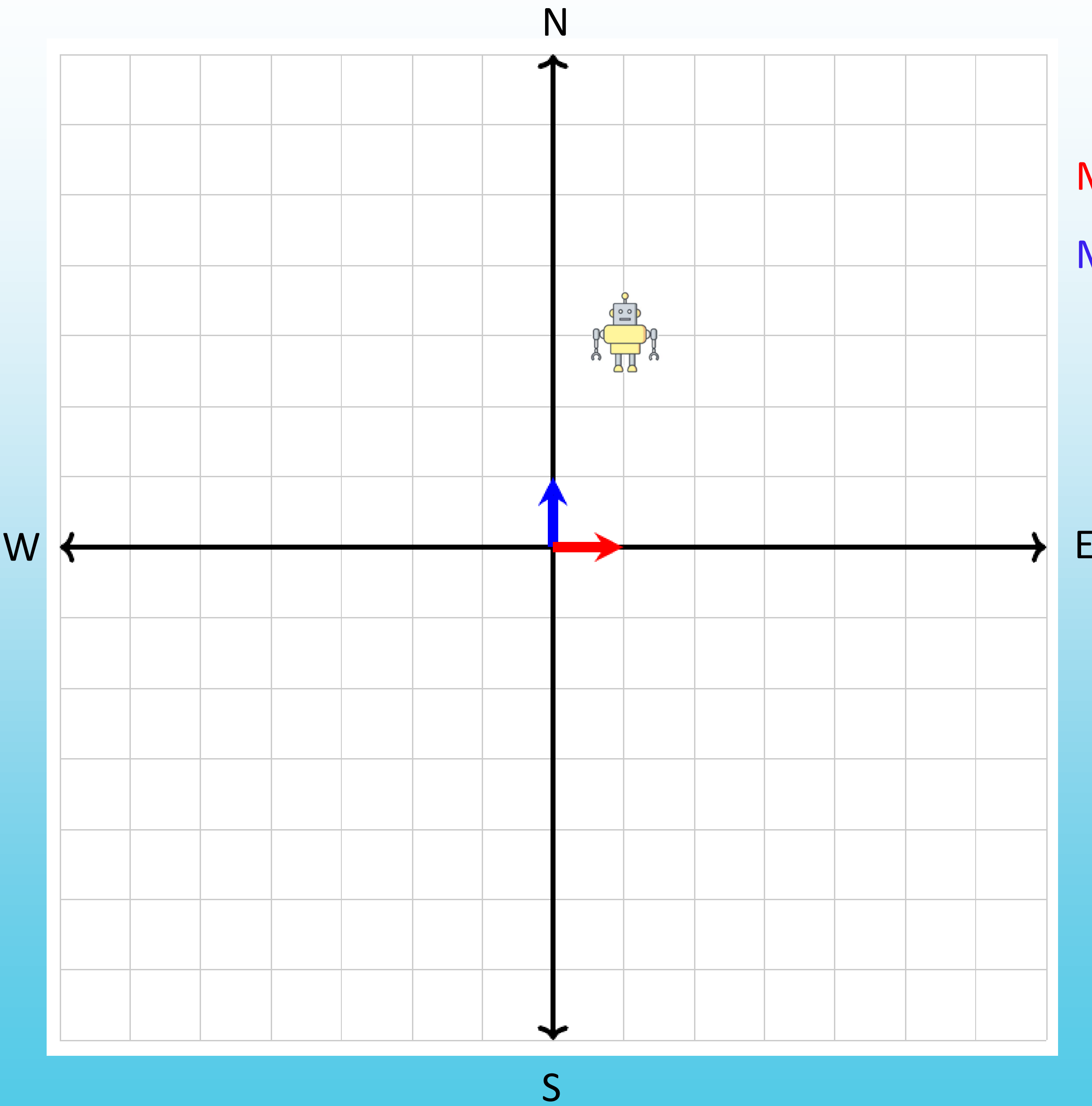


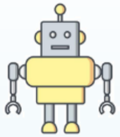


Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

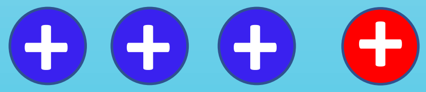
Some linear algebra...



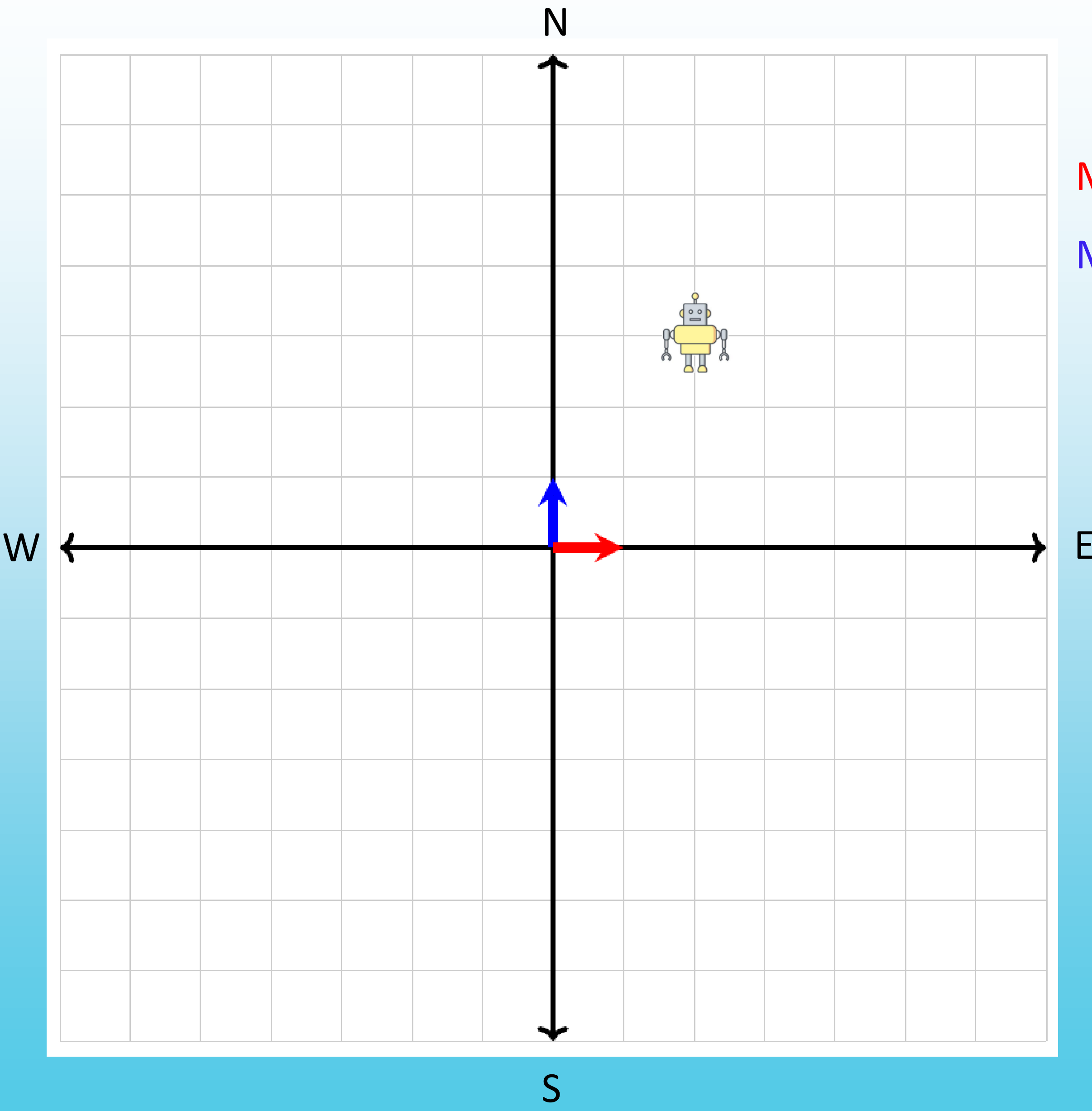


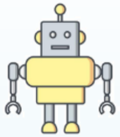
Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north



Some linear algebra…

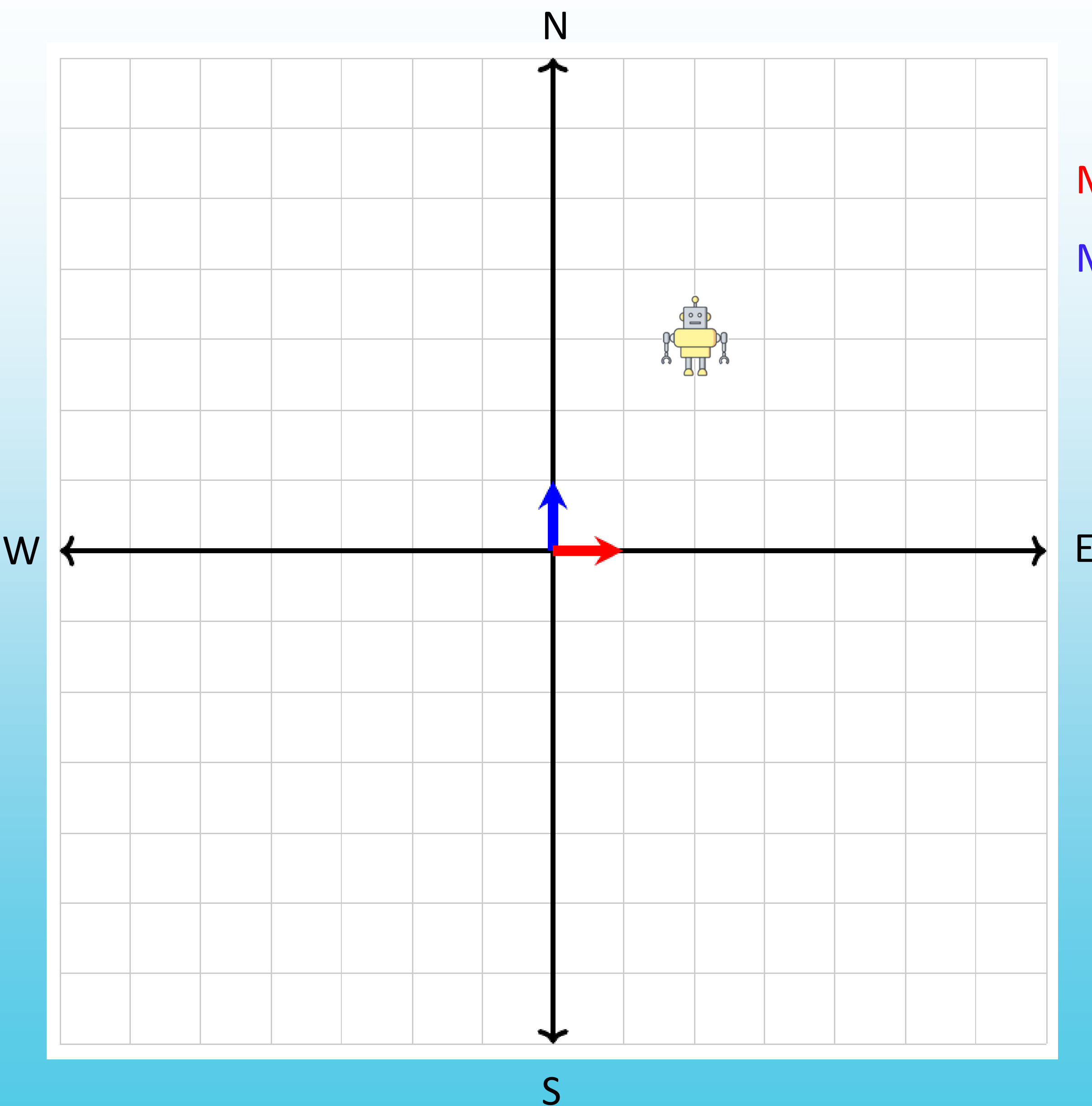


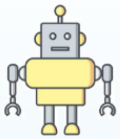


Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

Some linear algebra…



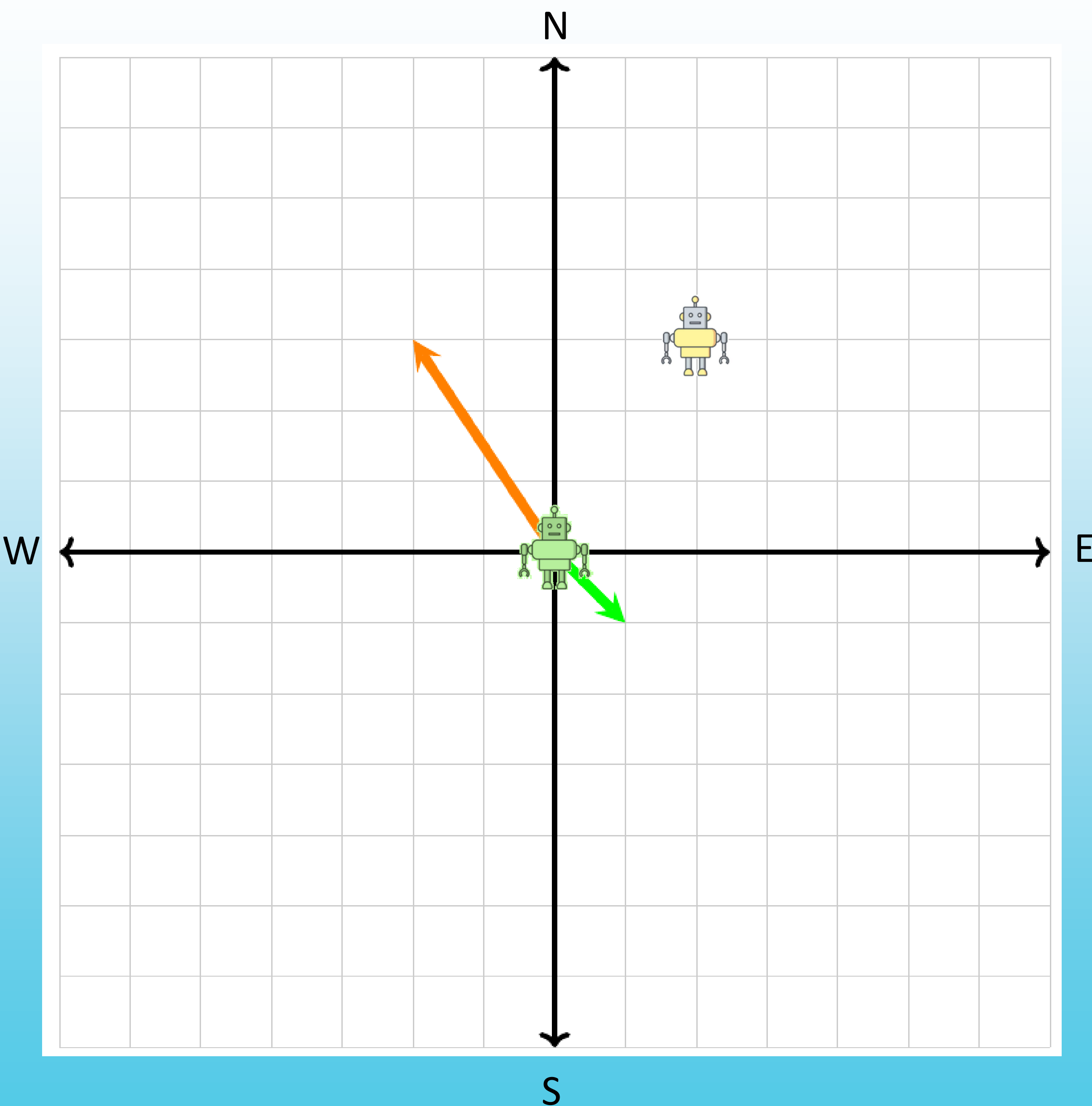


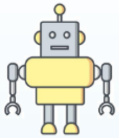
Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

2 (M1) + 3 (M2)

Some linear algebra…

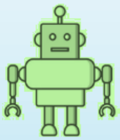




Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

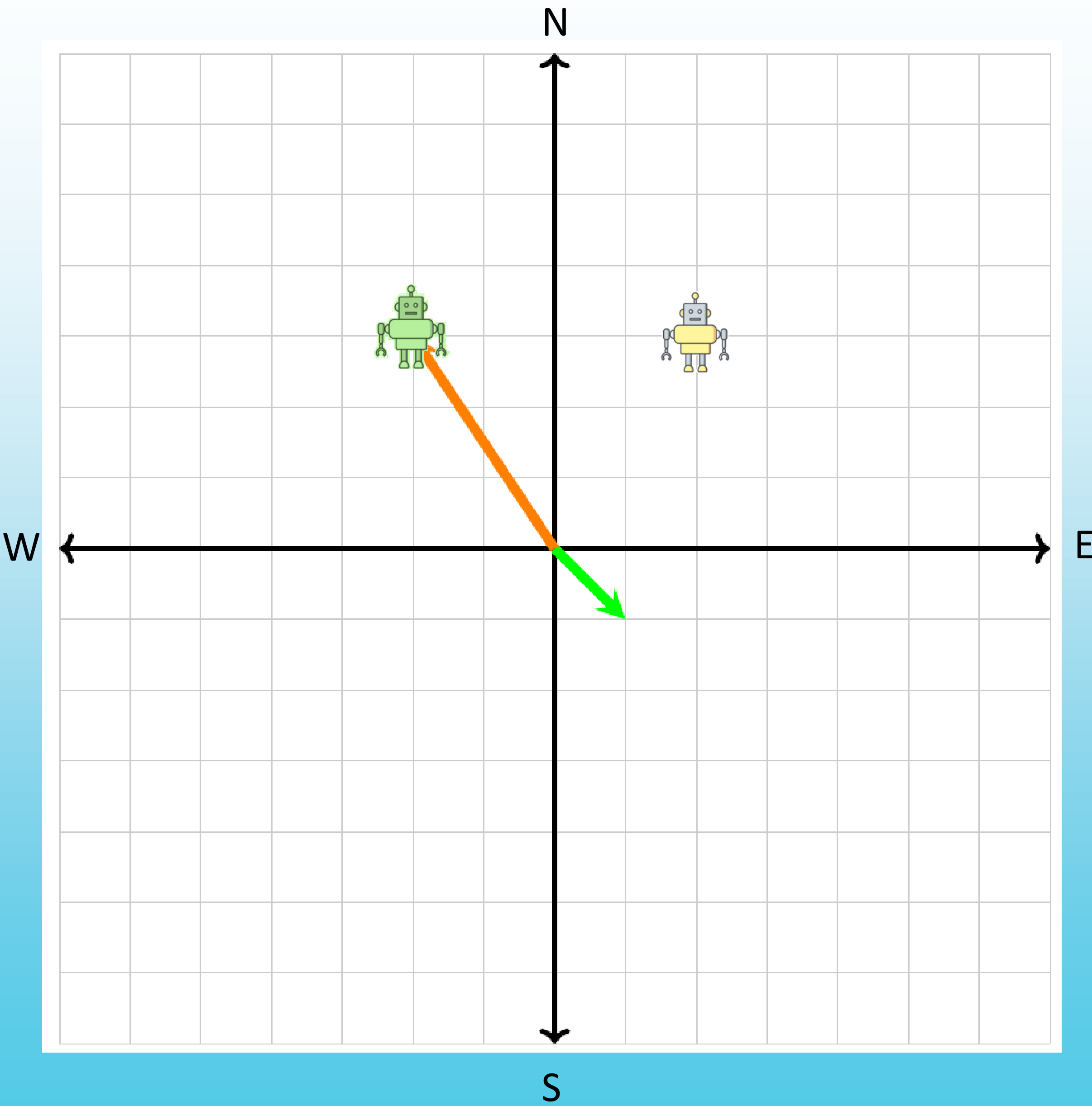
2 (M1) + 3 (M2)

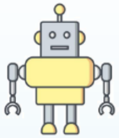


Movement 3 (M3): Go 1 step east & 1 step south

Movement 4 (M4): Go 2 steps west & 3 steps north

Some linear algebra...

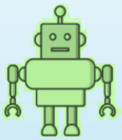




Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

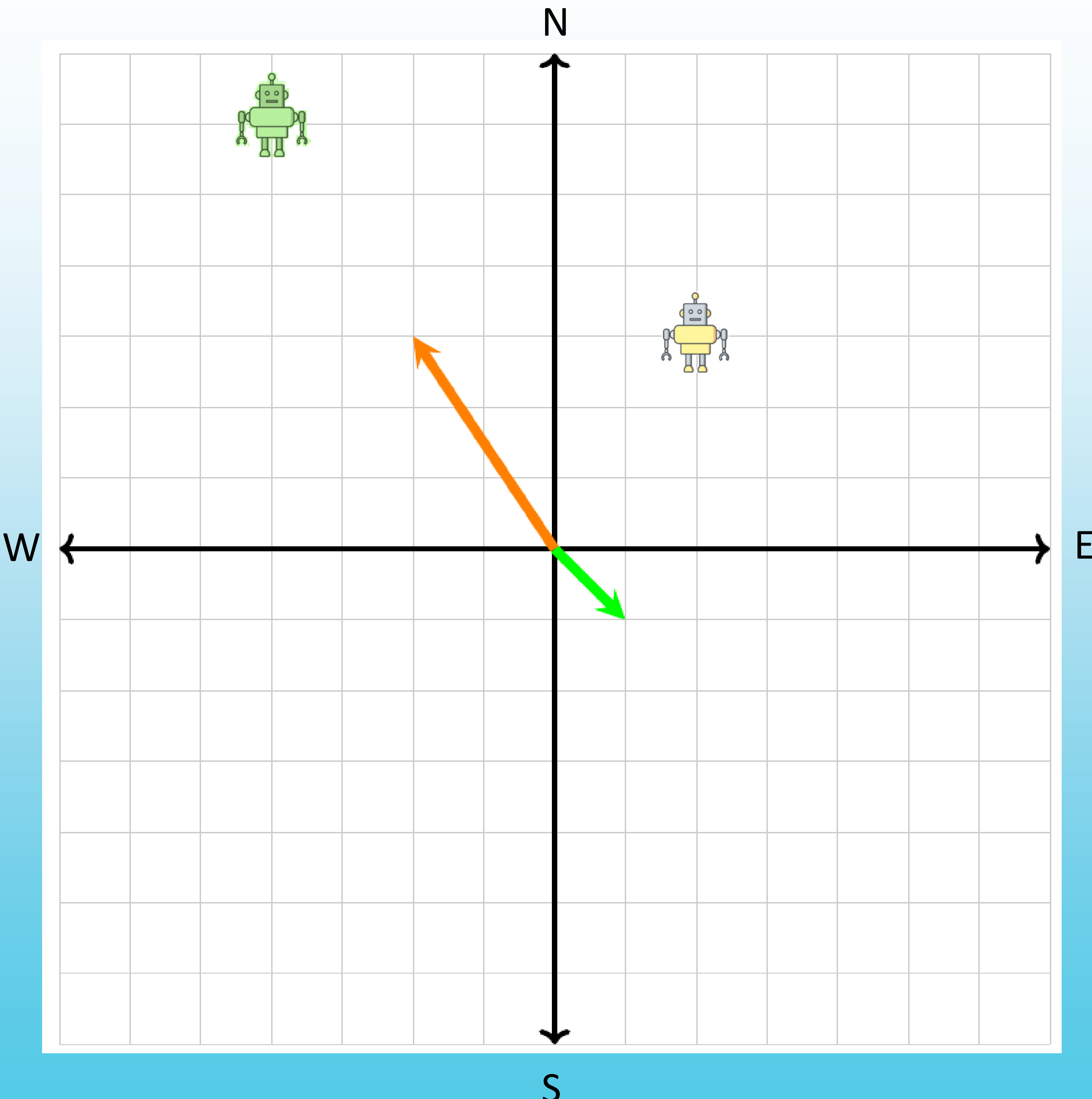
2 (M1) + 3 (M2)

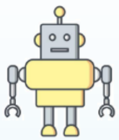


Movement 3 (M3): Go 1 step east & 1 step south

Movement 4 (M4): Go 2 steps west & 3 steps north

Some linear algebra...

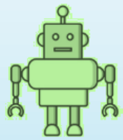




Movement 1 (M1): Go 1 step east

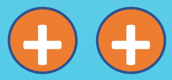
Movement 2 (M2): Go 1 step north

$2\,(M1) + 3\,(M2)$

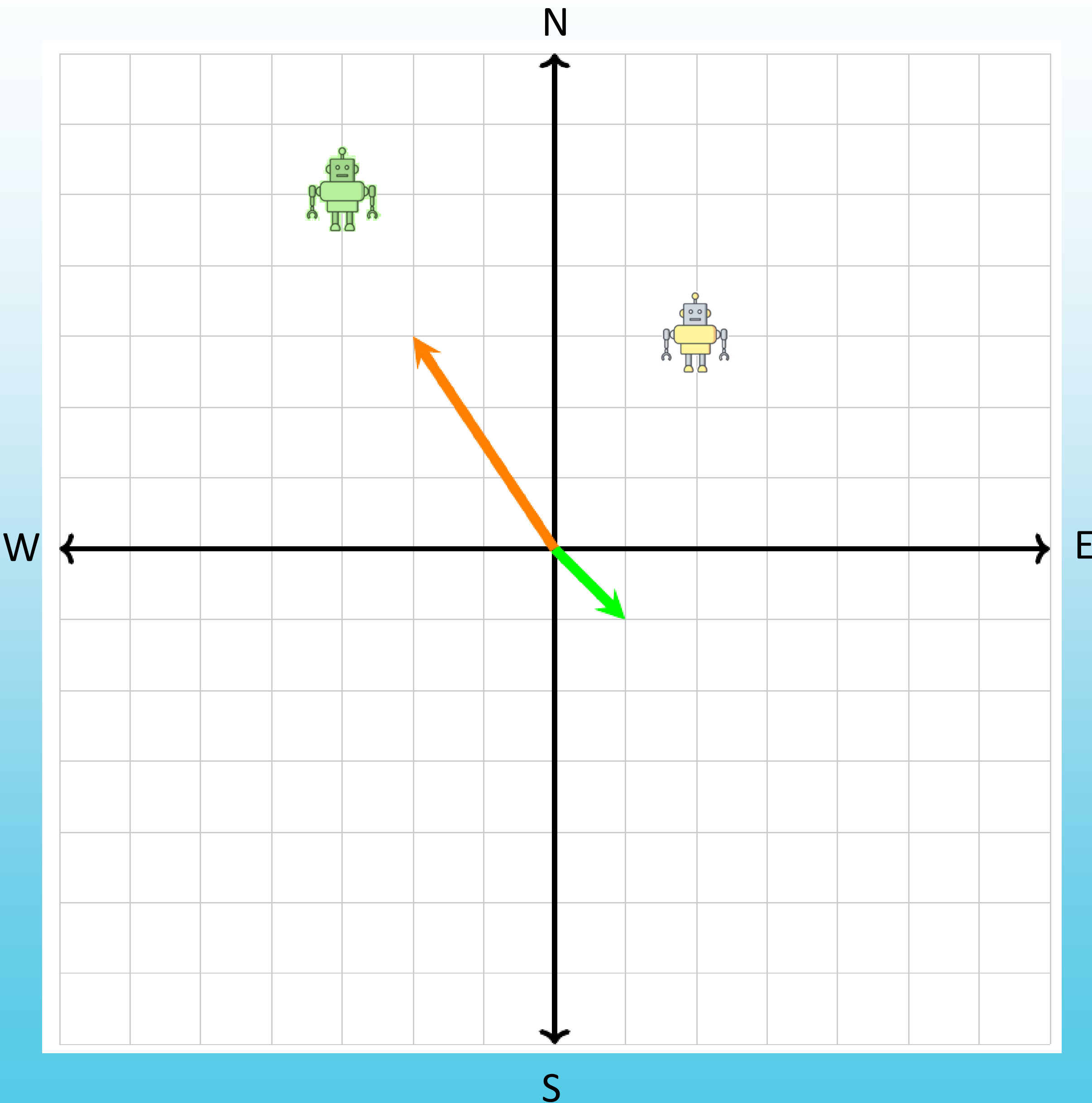


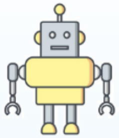
Movement 3 (M3): Go 1 step east & 1 step south

Movement 4 (M4): Go 2 steps west & 3 steps north



Some linear algebra…

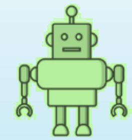




Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

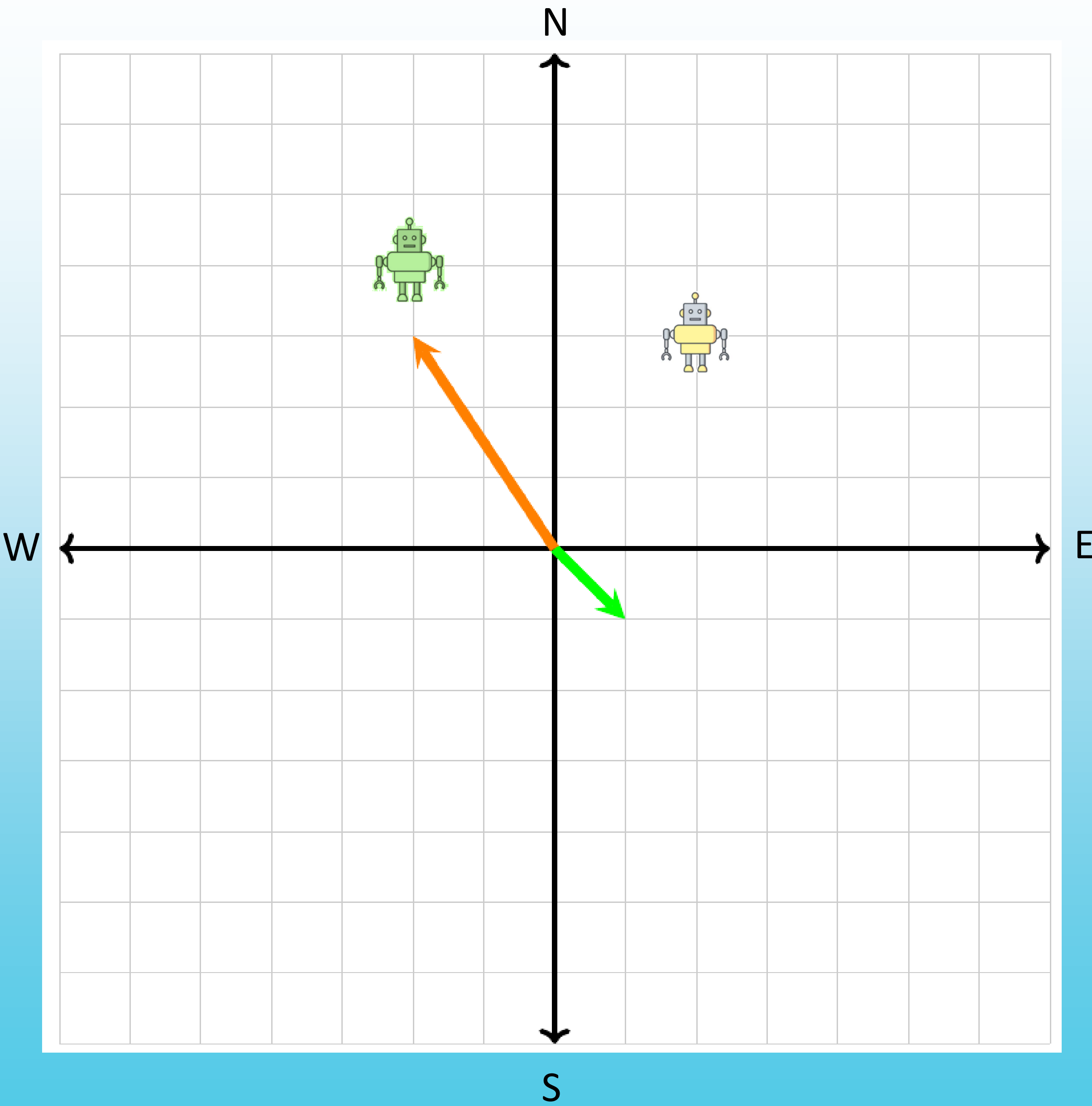
2 (M1) + 3 (M2)

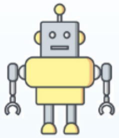


Movement 3 (M3): Go 1 step east & 1 step south

Movement 4 (M4): Go 2 steps west & 3 steps north

Some linear algebra...

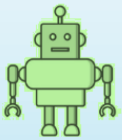




Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

$$2\,(M1) + 3\,(M2)$$



Movement 3 (M3): Go 1 step east & 1 step south

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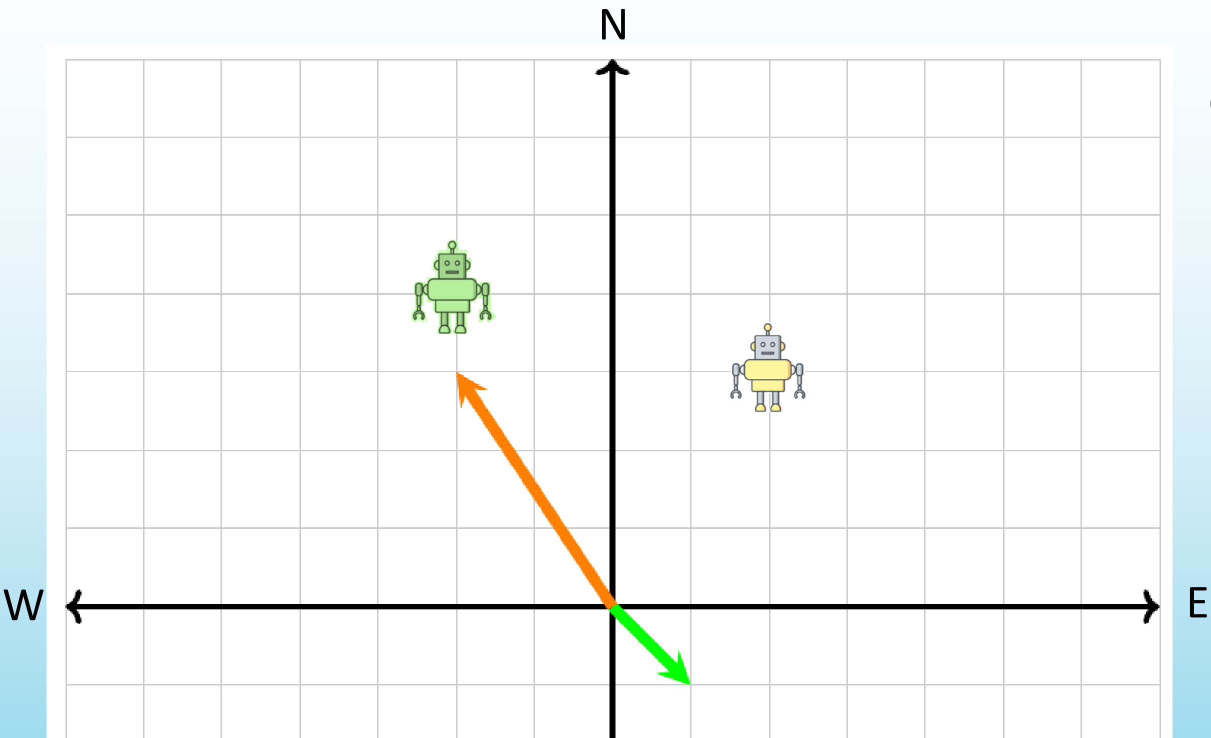
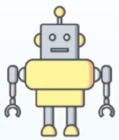
Some linear algebra…

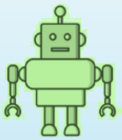




Movement 1 (M1): Go 1 step east

Movement 2 (M2): Go 1 step north

2 (M1) + 3 (M2)



Movement 3 (M3): Go 1 step east & 1 step south

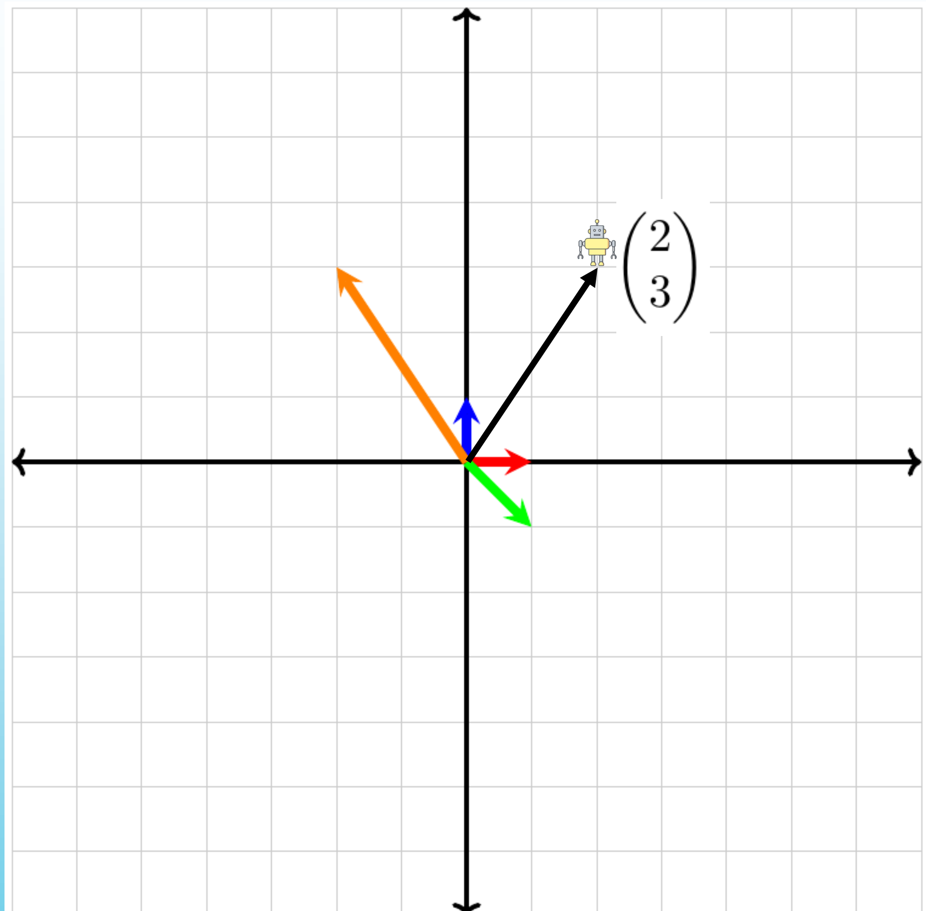
Movement 4 (M4): Go 2 steps west & 3 steps north

2 (M4) + 2 (M3)

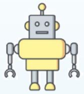
How can reach ?

? (M3) + ? (M4) = 3 (M2) + 2 (M1)

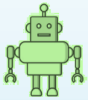
Some linear algebra…a bit more serious



Vector Notation:

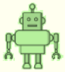


$$M_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$M_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$2M_1 + 3M_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$



Find numbers a and b such that

$$aM_3 + bM_4 = a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Matrix * Vector = Vector

$$\underbrace{\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 + 3\cdot 2 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 12 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

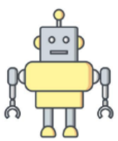
$$12M_3 + 5M_4 = 12\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 5\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 12-10 \\ -12+15 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Basis change

These two sets of „movements“ are examples of bases for the 2-dimensional space.

- Every point can be reached
- There is a unique way to reach a point

Basis 1 (Standard basis)



$$M_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{2}M_1 + \underline{3}M_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

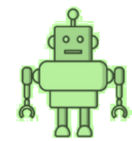
Change of basis

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \underline{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} = \underline{\begin{pmatrix} 12 \\ 5 \end{pmatrix}}$$

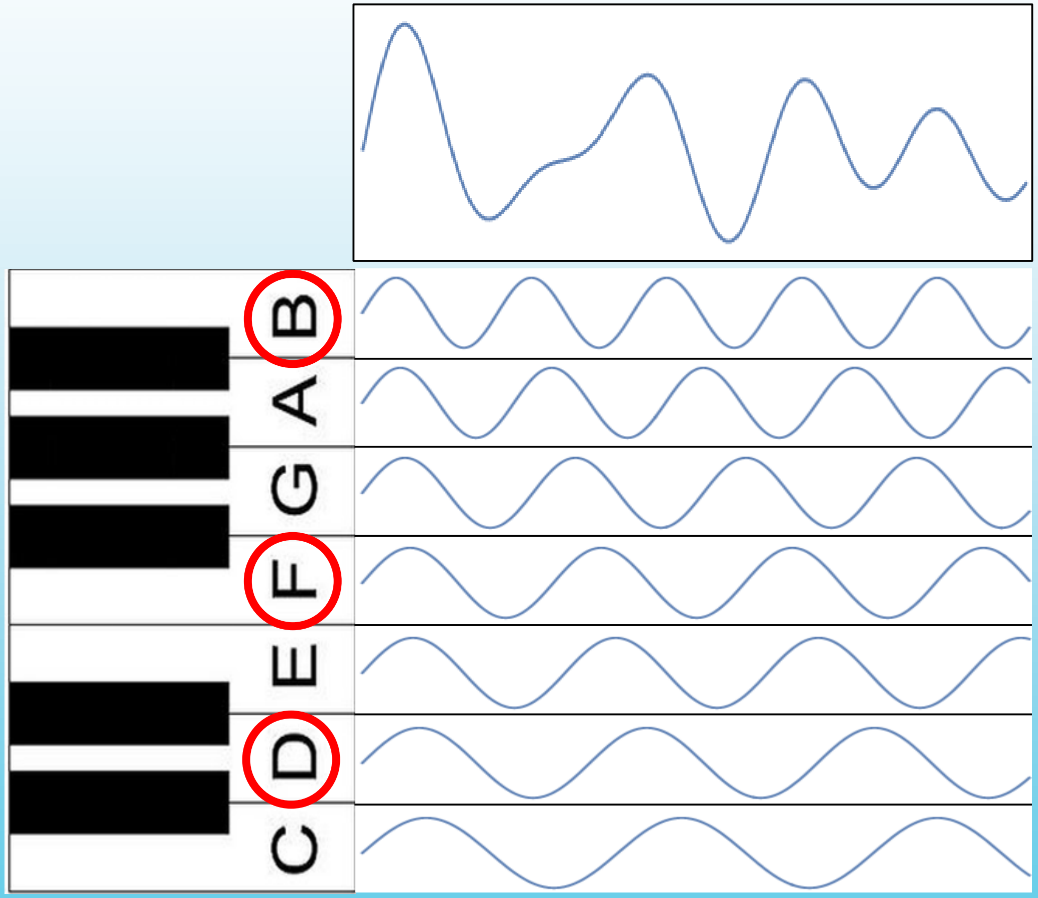
Basis 2



$$M_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\underline{12}M_3 + \underline{5}M_4 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

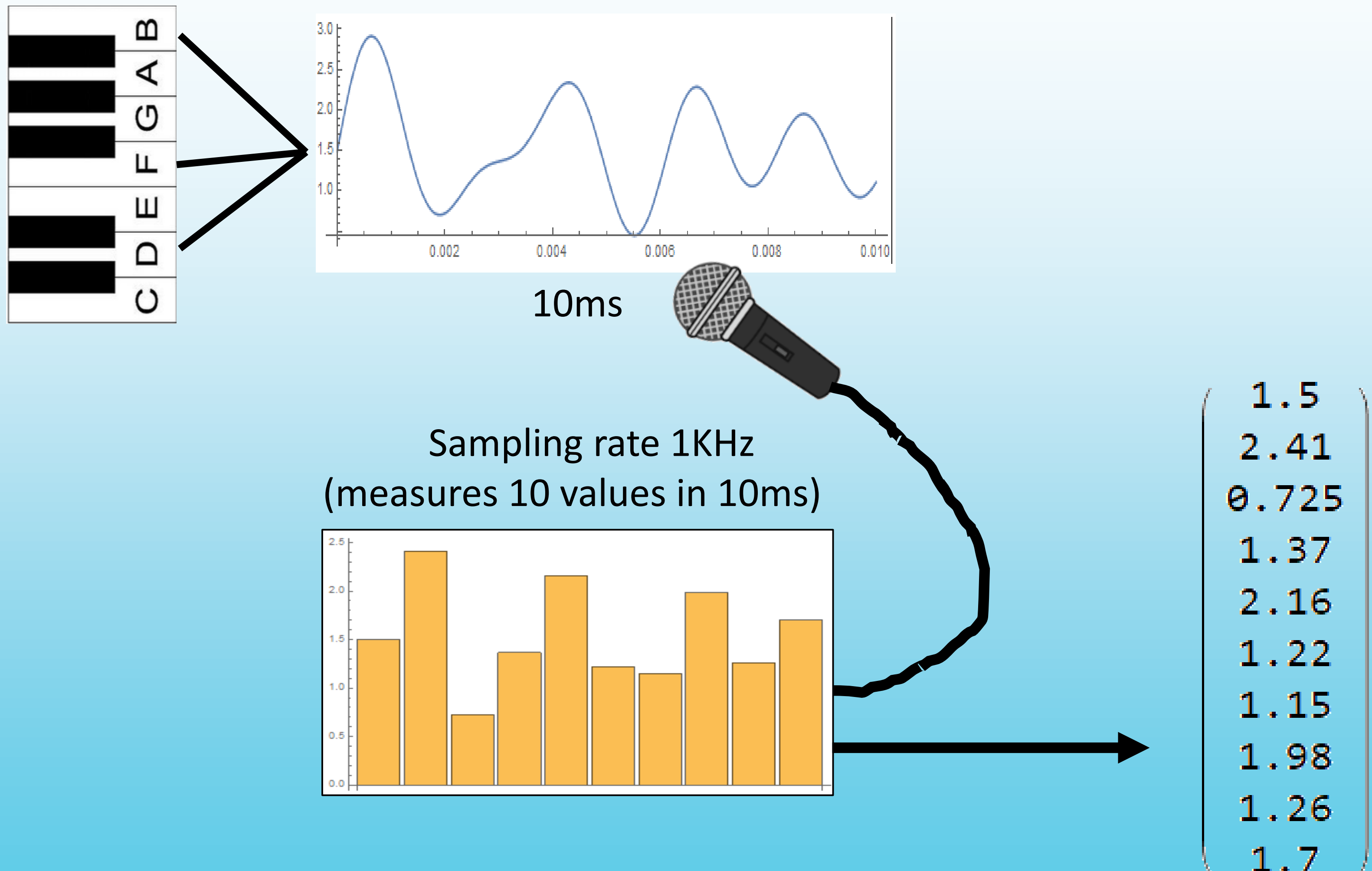
Back to SingStar



D + F + B

How to we get this result using math?

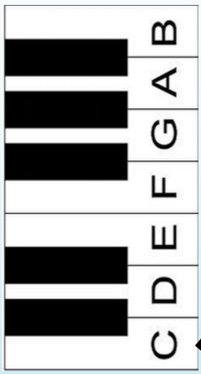
Back to SingStar: Recording with a microphone

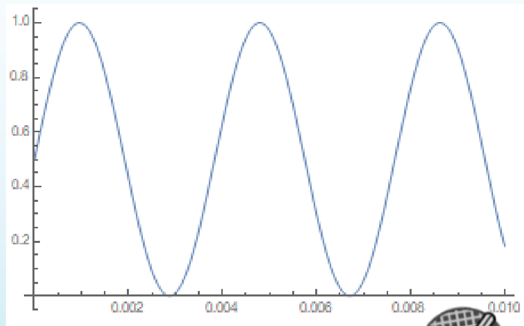


Soundwave → Vector

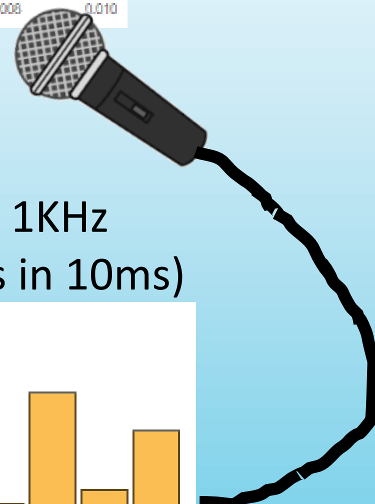
Back to SingStar: Recording with a microphone

But we can do that for each Key seperately first!

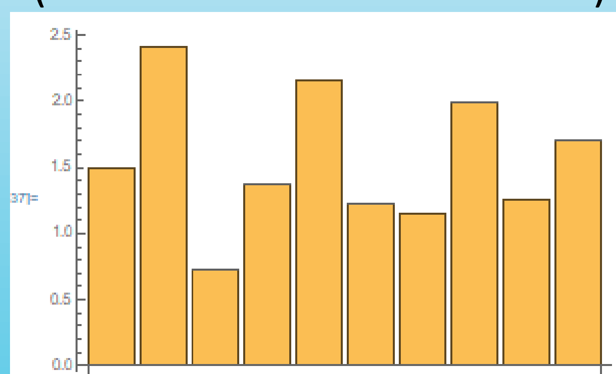




10ms



Sampling rate 1KHz
(measures 10 values in 10ms)



$$\begin{pmatrix} 0.5 \\ 0.999 \\ 0.431 \\ 0.0107 \\ 0.636 \\ 0.97 \\ 0.299 \\ 0.0574 \\ 0.763 \\ 0.906 \end{pmatrix}$$

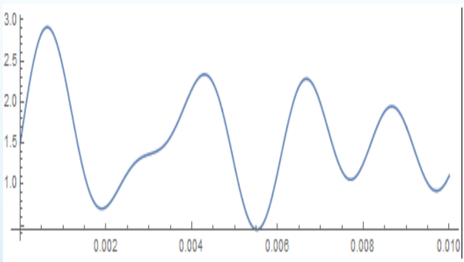
Soundwave → Vector

SingStar: Just Linear Algebra

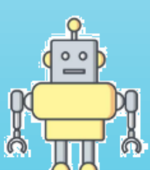
We obtain the same question as before

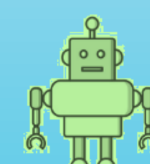
Find numbers a and b such that

$$aM_3 + bM_4 = a\begin{pmatrix}1\\-1\end{pmatrix} + b\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$$



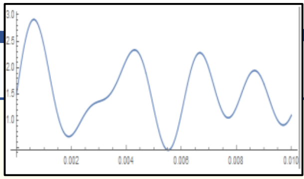
$$\begin{pmatrix}1.5\\2.41\\0.725\\1.37\\2.16\\1.22\\1.15\\1.98\\1.26\\1.7\end{pmatrix} = C\begin{pmatrix}0.5\\0.999\\0.431\\0.0107\\0.636\\0.97\\0.299\\0.0574\\0.763\\0.906\end{pmatrix} + D\begin{pmatrix}0.5\\0.982\\0.243\\0.155\\0.941\\0.609\\0.000632\\0.657\\0.915\\0.121\end{pmatrix} + \dots$$





Find numbers C,D,... such that this equations holds

SingStar: Just Linear Algebra

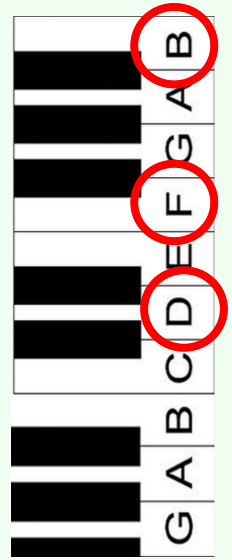


Solution

$$\begin{pmatrix} 1.5 \\ 2.41 \\ 0.725 \\ 1.37 \\ 2.16 \\ 1.22 \\ 1.15 \\ 1.98 \\ 1.26 \\ 1.7 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.52 & 0.68 & 0.82 & 0.91 & 0.94 & 0.98 & 1. & 1. & 0.99 & 0.97 \\ 0.46 & 0.16 & 0.01 & 0.026 & 0.081 & 0.24 & 0.43 & 0.53 & 0.68 & 0.82 \\ 0.57 & 0.95 & 0.94 & 0.65 & 0.46 & 0.16 & 0.011 & 0.0014 & 0.078 & 0.25 \\ 0.41 & 0.00099 & 0.3 & 0.8 & 0.96 & 0.94 & 0.64 & 0.45 & 0.16 & 0.0089 \\ 0.61 & 0.98 & 0.36 & 0.00025 & 0.1 & 0.61 & 0.97 & 1. & 0.79 & 0.42 \\ 0.37 & 0.11 & 0.91 & 0.78 & 0.42 & 0.00063 & 0.3 & 0.58 & 0.95 & 0.94 \\ 0.65 & 0.74 & 0.0017 & 0.68 & 0.97 & 0.66 & 0.057 & 0.0077 & 0.38 & 0.88 \\ 0.33 & 0.44 & 0.86 & 0.017 & 0.13 & 0.92 & 0.76 & 0.4 & 0.00099 & 0.32 \\ 0.69 & 0.38 & 0.44 & 0.89 & 0.38 & 0.12 & 0.91 & 0.99 & 0.44 & 0.00025 \end{pmatrix} \begin{pmatrix} 1. \\ -2.45 \times 10^{-14} \\ 1.07 \times 10^{-13} \\ 1. \\ 7.59 \times 10^{-13} \\ 1. \\ 1.27 \times 10^{-12} \\ -1.07 \times 10^{-12} \\ 2.76 \times 10^{-13} \\ -4.59 \times 10^{-14} \end{pmatrix}$$

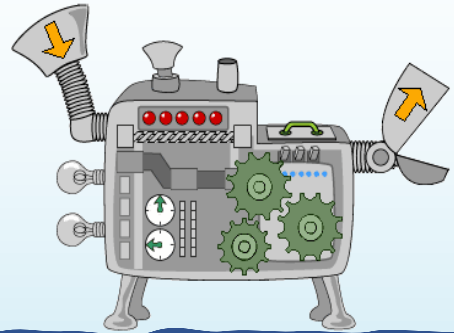
Interpretation of the solution:

$$\begin{pmatrix} 1. \\ -2.45 \times 10^{-14} \\ 1.07 \times 10^{-13} \\ 1. \\ 7.59 \times 10^{-13} \\ 1. \\ 1.27 \times 10^{-12} \\ -1.07 \times 10^{-12} \\ 2.76 \times 10^{-13} \\ -4.59 \times 10^{-14} \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



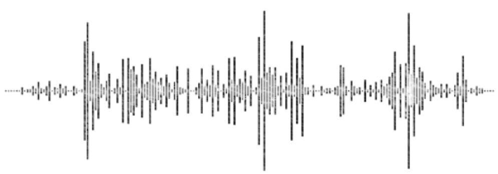
The keys D,F and B were played with the same volume (all values are the same)

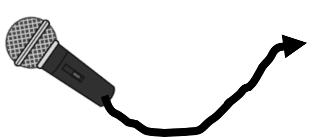
SingStar: How does it work?



SingStar machine

Soundwave





Vector
(of pressure data)

$$\begin{pmatrix} 1.5 \\ 2.41 \\ 0.725 \\ 1.37 \\ 2.16 \\ 1.22 \\ 1.15 \\ 1.98 \\ 1.26 \\ 1.7 \end{pmatrix}$$

Matrix multiplication
(basis change)

Discrete Fourier transform

Vector
(of amplitudes, i.e.
Intensity in each frequency)

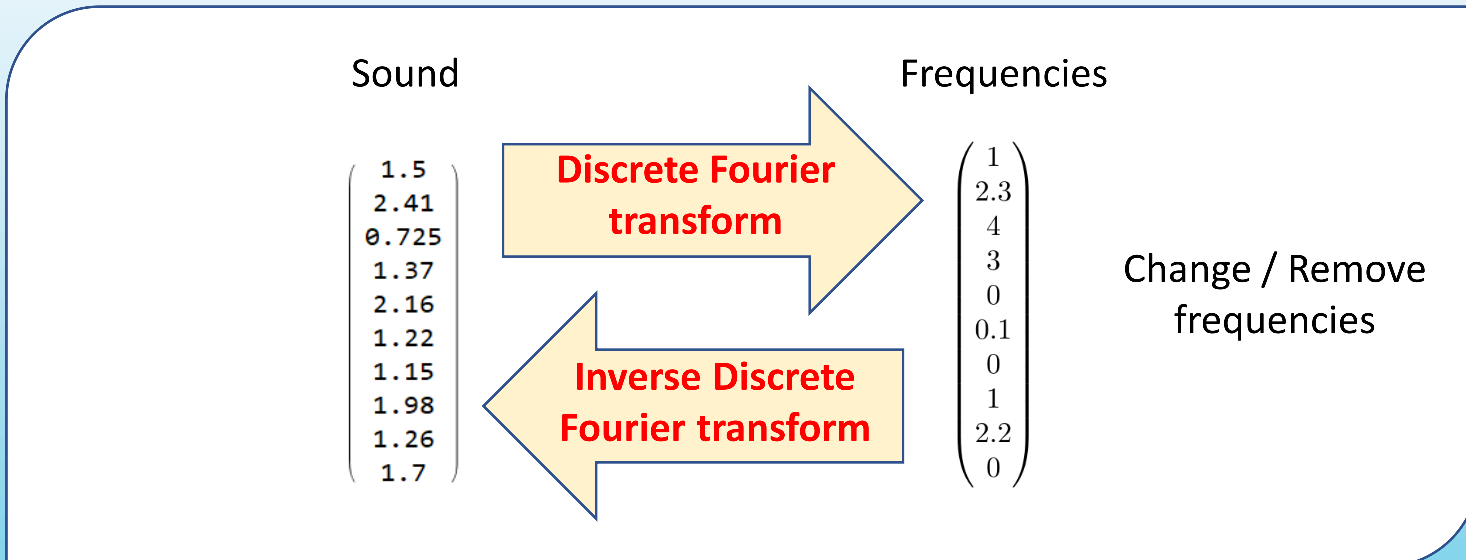
$$\begin{pmatrix} 1 \\ 2.3 \\ 4 \\ 3 \\ 0 \\ 0.1 \\ 0 \\ 1 \\ 2.2 \\ 0 \end{pmatrix}$$

The highest values gives the current pitch

Fourier transform

The (discrete) Fourier transform has various applications

- Digital filter

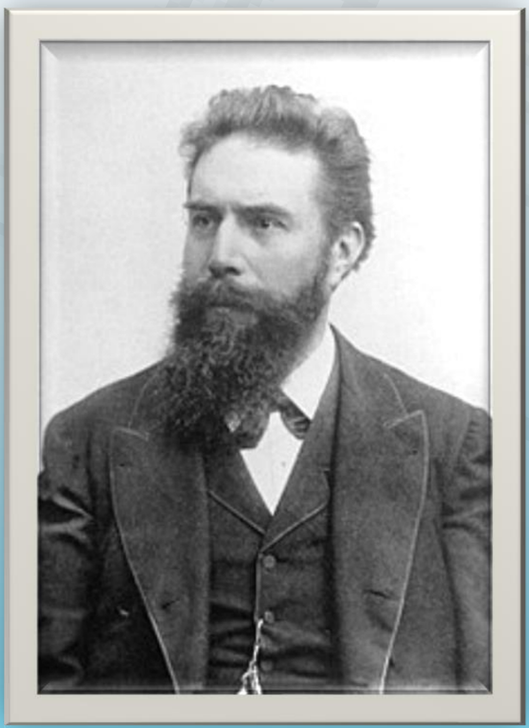


- Image processing
- Data compressions (JPEG)
- Appears in various areas of mathematics and physics

A little bit history

Today: Friday 8th November, 2019

124 years ago: Friday 8th November, 1895

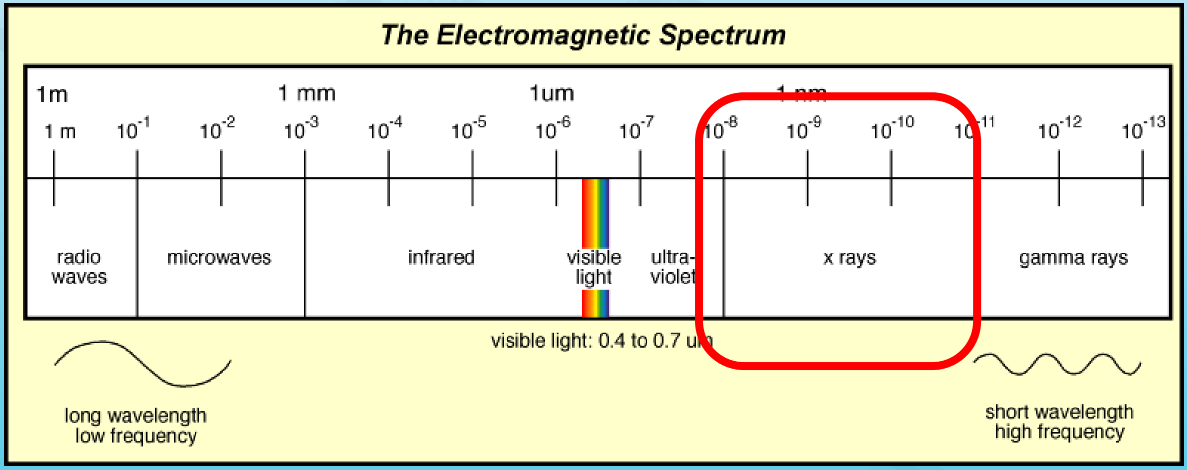


Wilhelm Conrad Röntgen

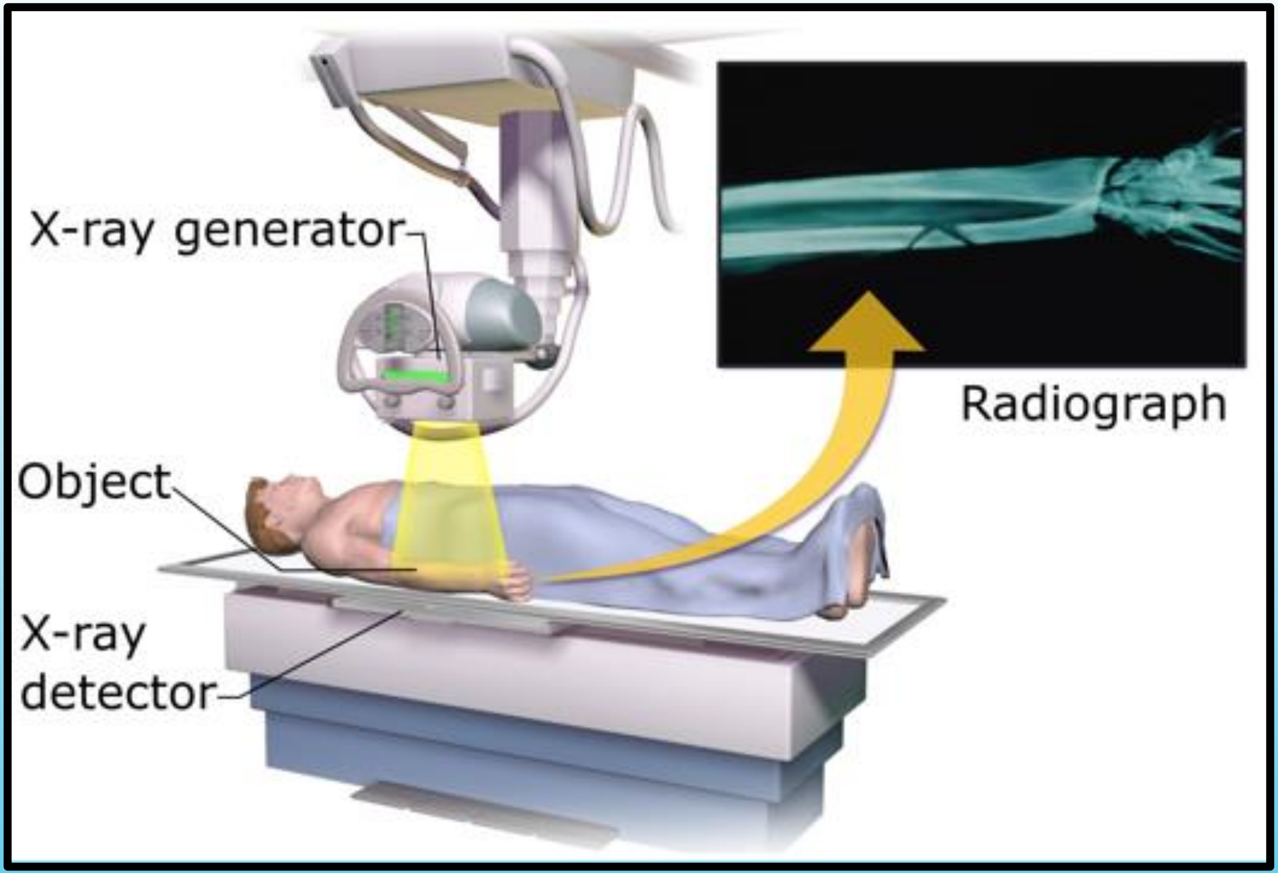
1845 - 1923

Today 124 years ago W. C. Roentgen discovered X-Rays (Röntgenstrahlen, レントゲン線)

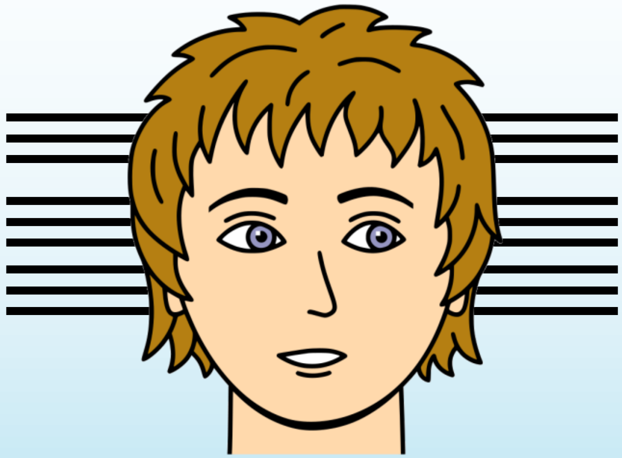
For this discovery he obtained the first Nobel Prize in Physics (1901)

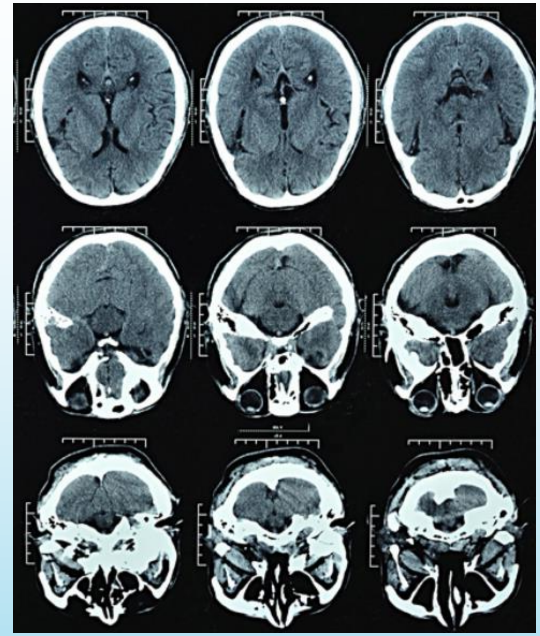


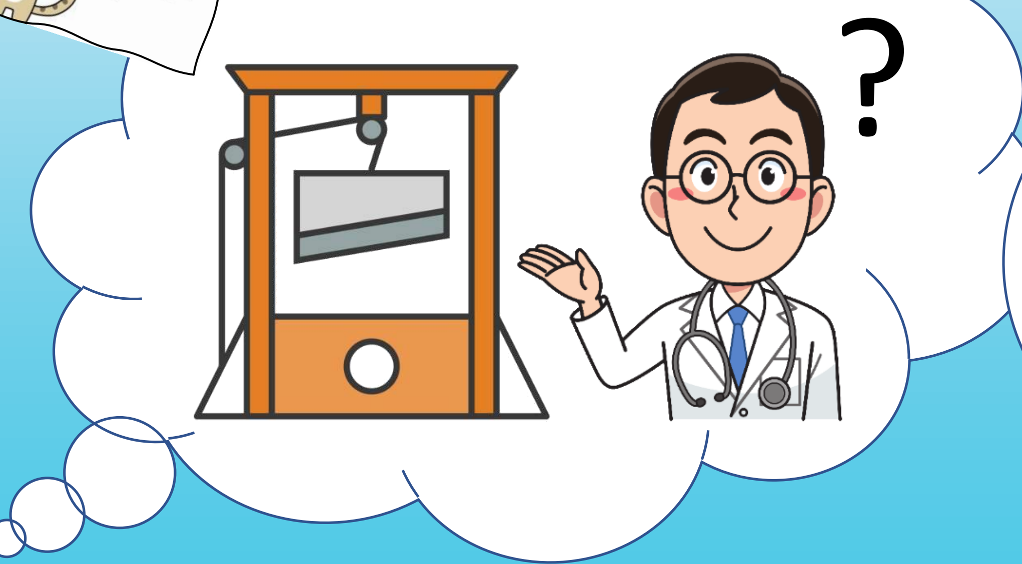
X-rays

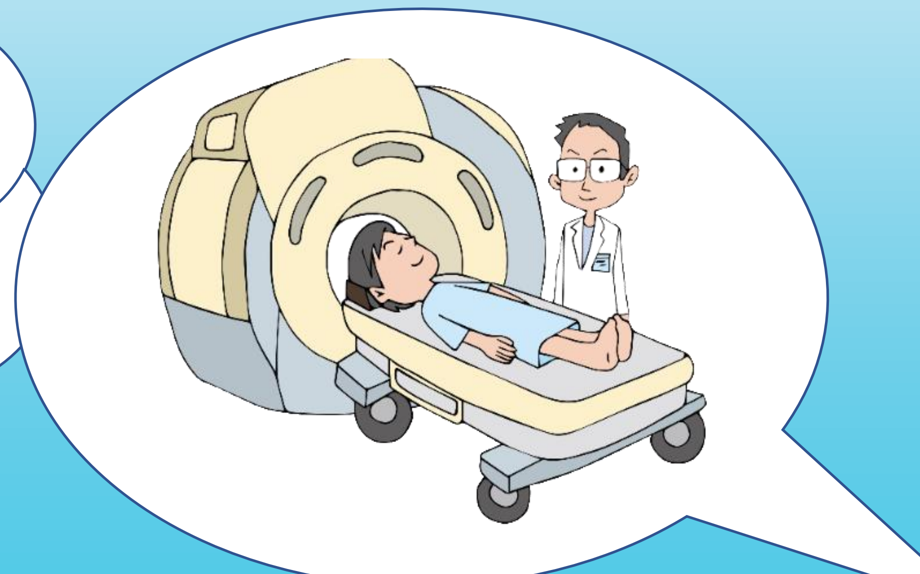


Different point of view

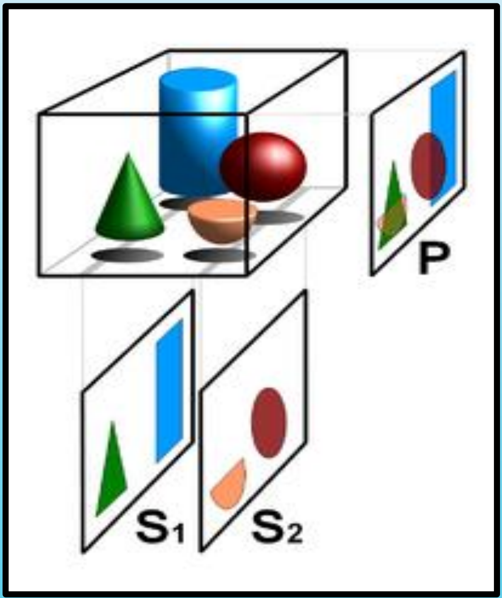








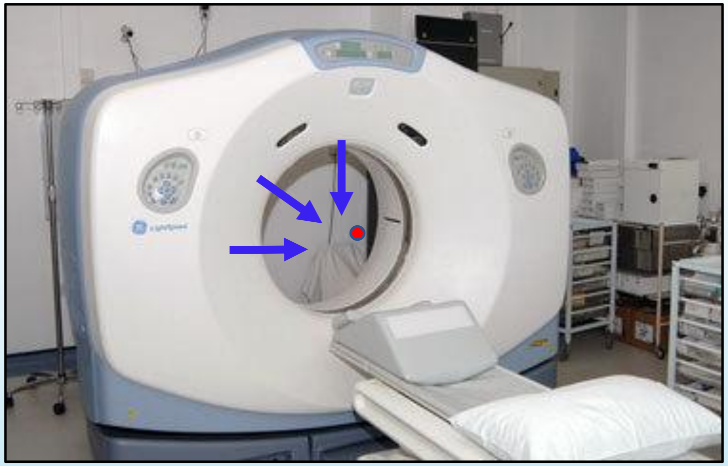
Computed tomography scan (CT Scan)

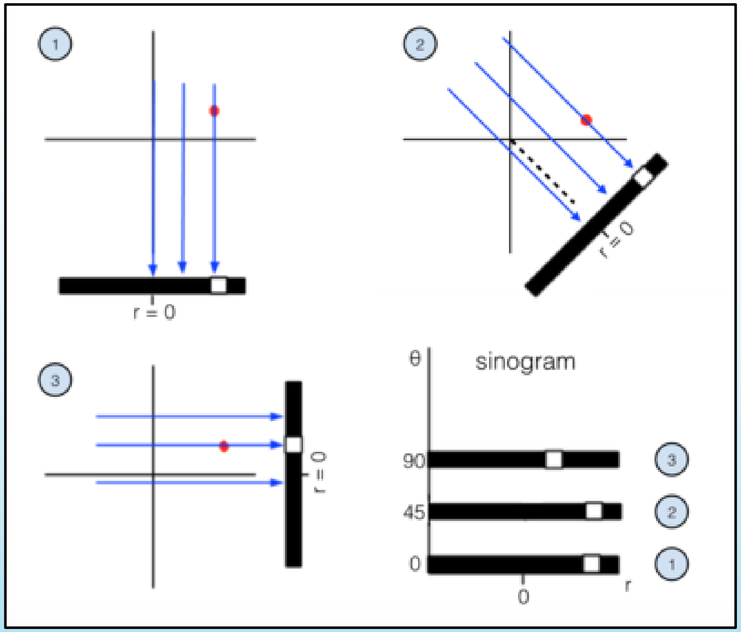


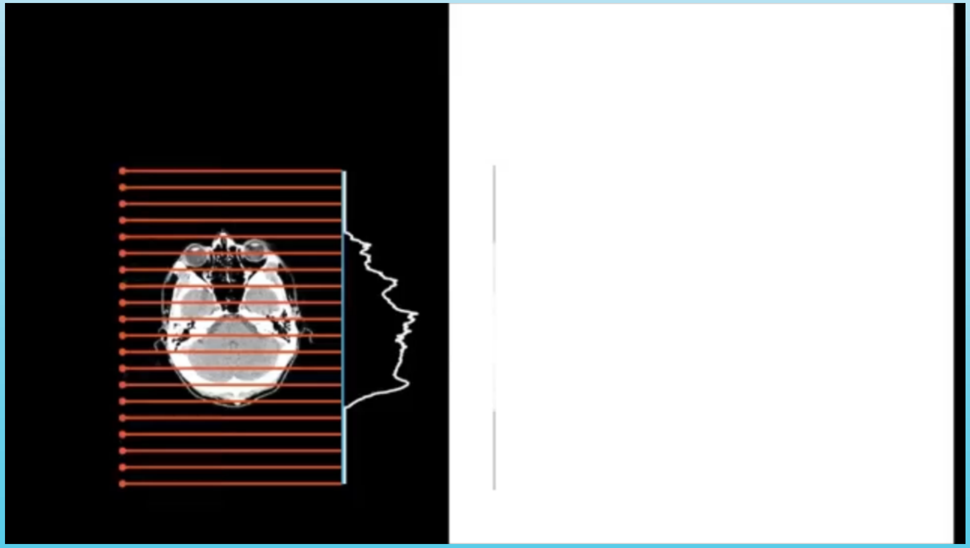
Basic principle of **tomography**
(tomos = slice/section)

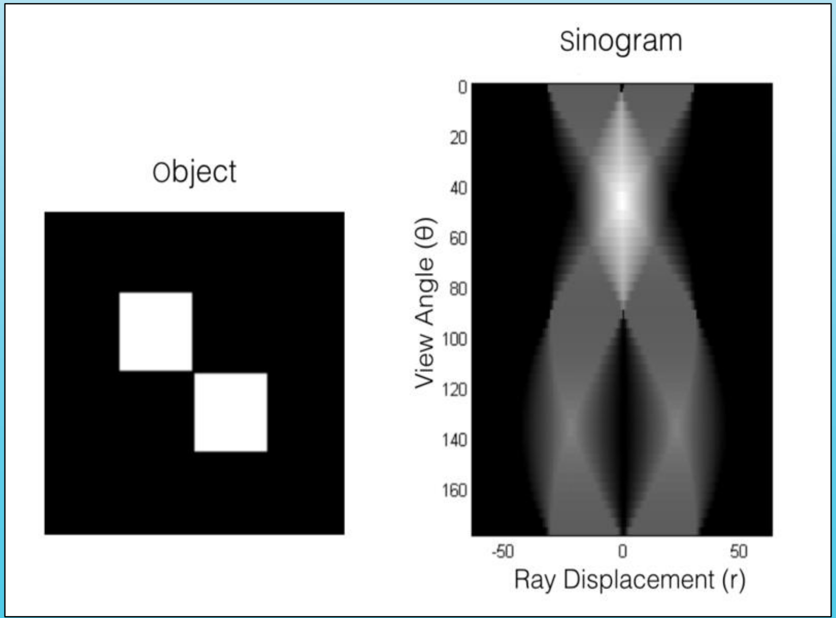
CT Scanner
(without cover)

X-Ray from different angles: Sinogram





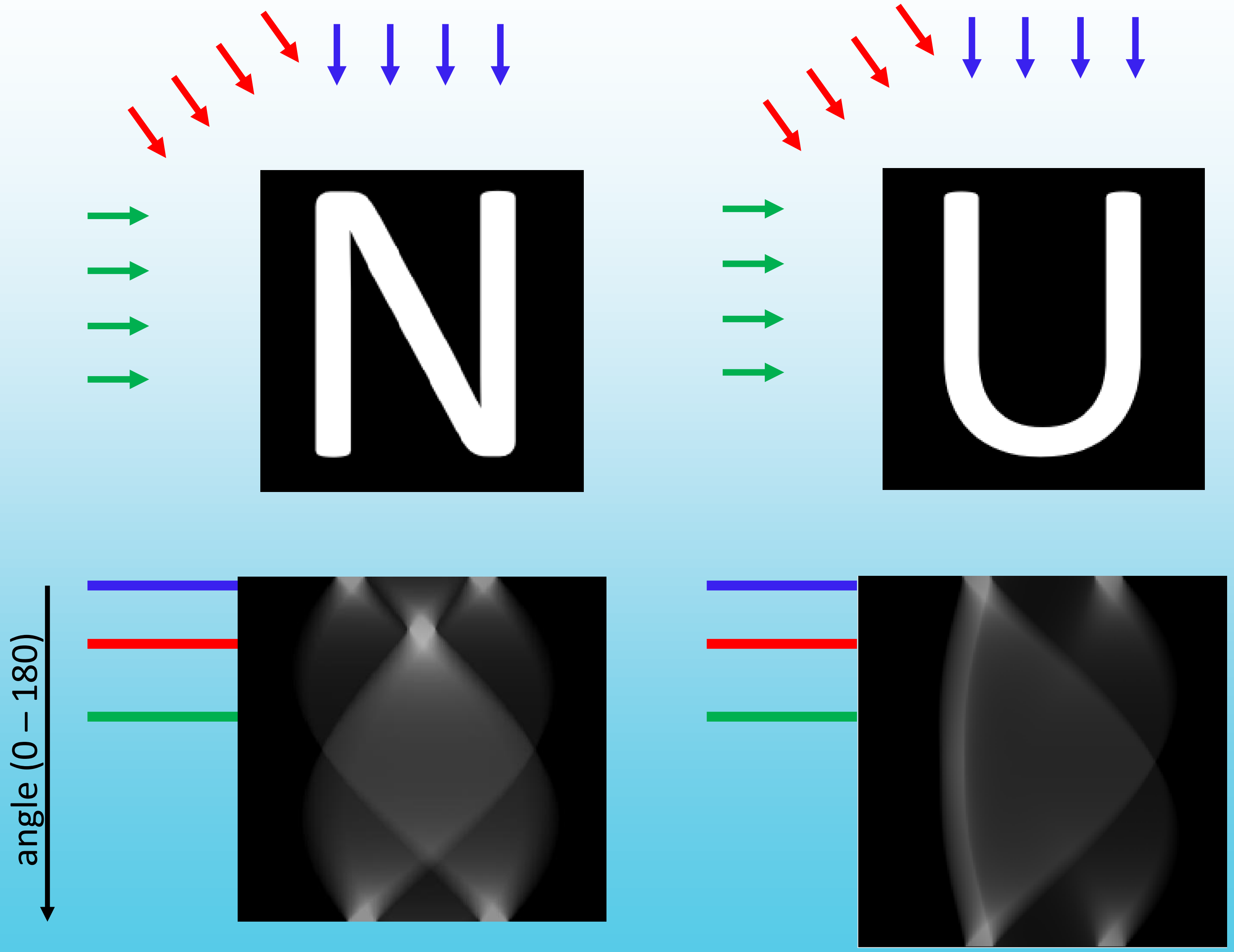




X-Ray from different angles: Sinogram

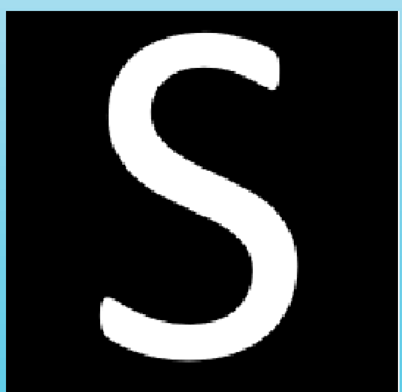
Original object

Sinogram
(taking X-ray
from different
angels)



Sinogram: Can you invert it?

Is this the Sinogram of S or G?



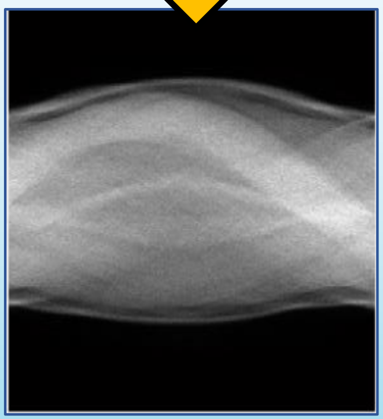
Yes

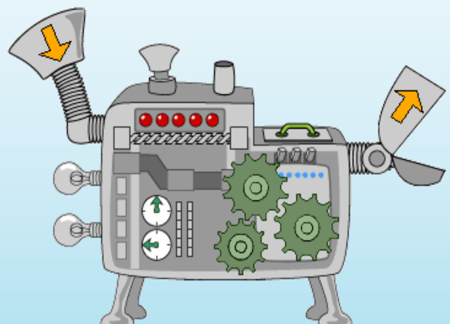
No

Sinogram: Can you invert it?

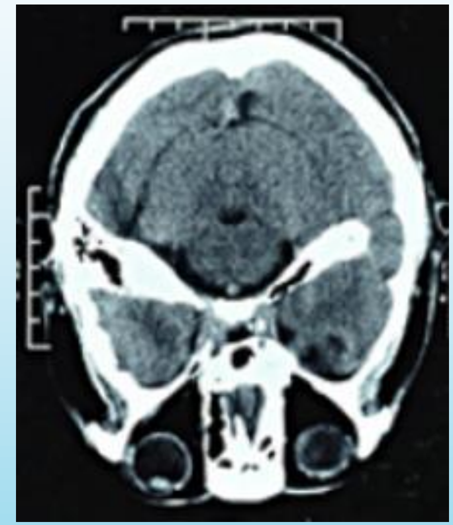
Correct: Yes

CT Scan – How does it work?





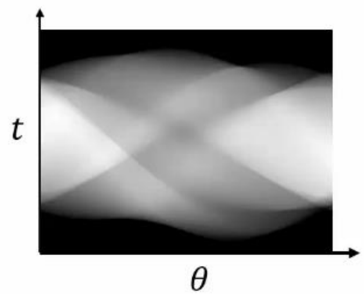
CT Scan

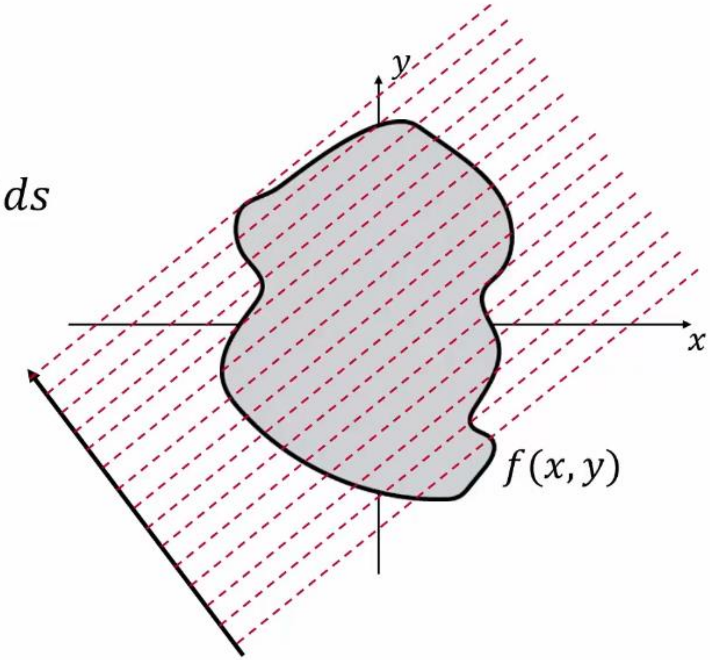


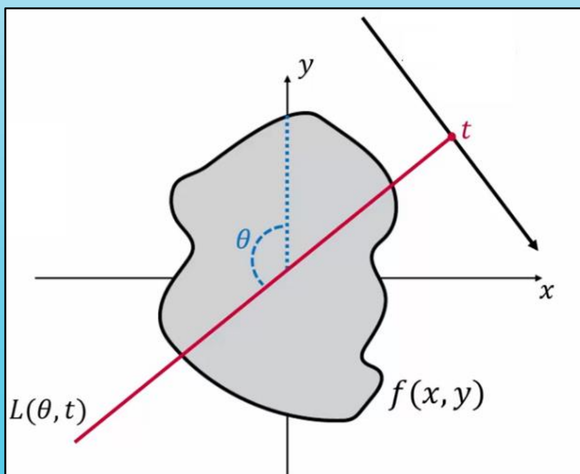
Radon transform

Radon transform (sorry a little bit math)

$$\mathcal{R}f(\theta,t) = \int_{L(\theta,t)} f(x,y)\,ds$$



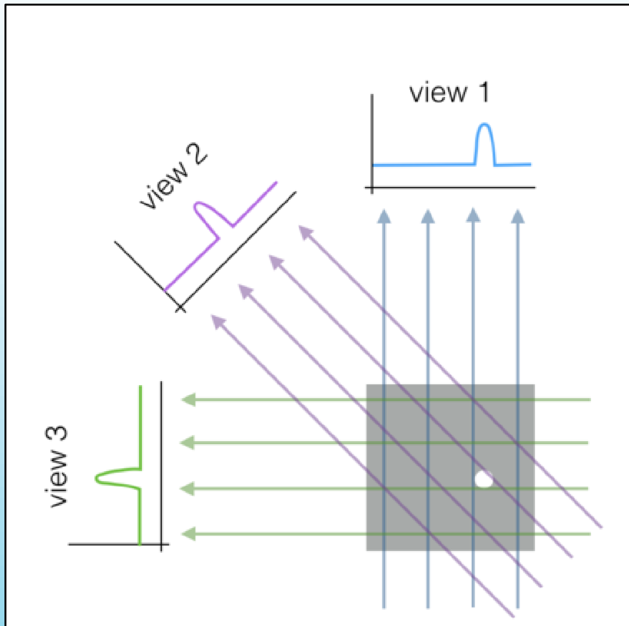


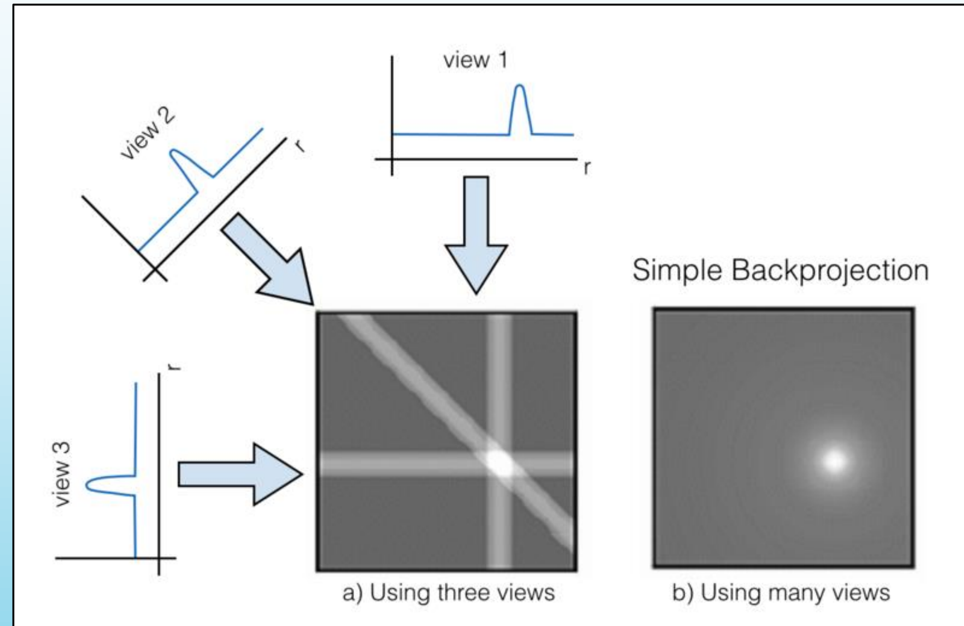


This transform can be inverted by using the Fourier transform and the „Projection slice theorem“.

Backprojection: From Sinogram to the original object

- One easy way: Simple backprojection

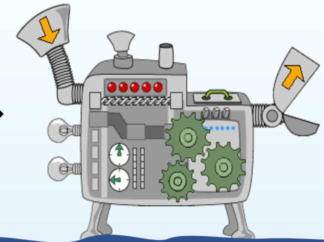




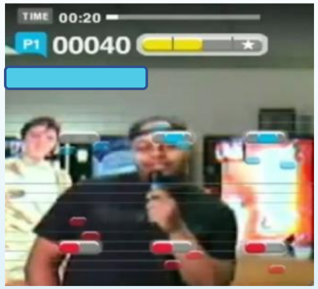
Results are blurry!

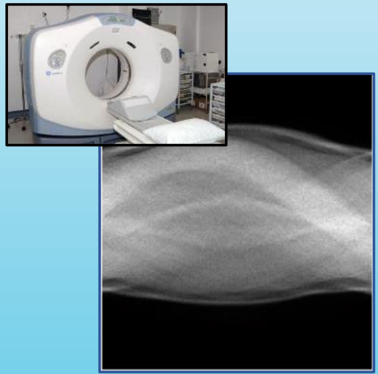
In reality **filtered backprojection** is used
(uses discrete Fourier transform)

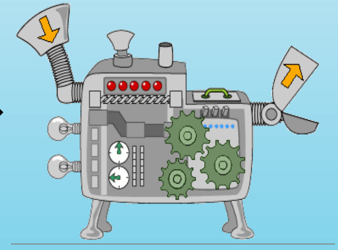
Summary: Math gives magic machines



Discrete Fourier transform







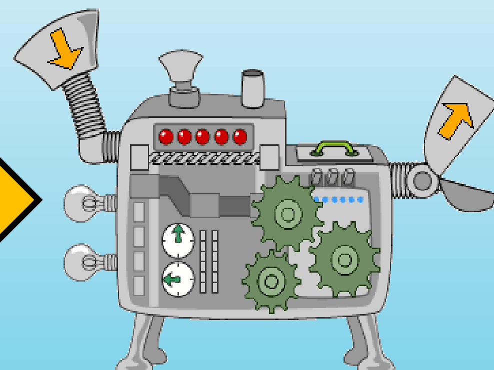
Filtered backprojection

Some problems are too hard..

In reality, we also have problems where it is really hard/impossible to create a magic machine.

Example: Ramen classification

Input
Picture of Ramen

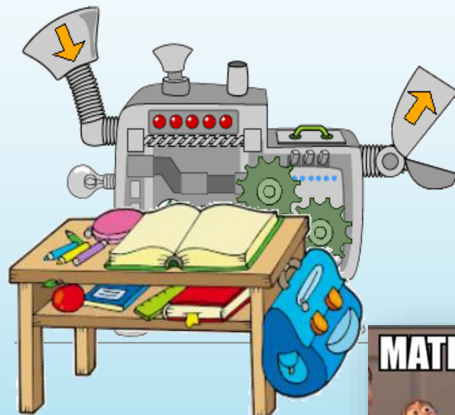


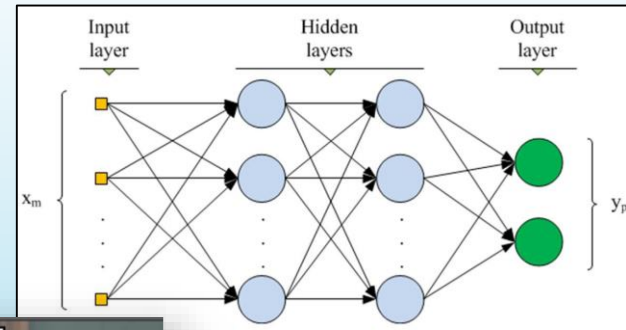
?????

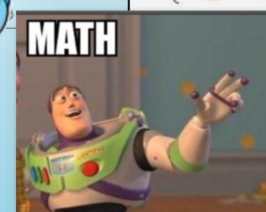
Output
What type of Ramen is it?
Taiwan Ramen?
Tonkotsu Ramen?
...

Machine learning

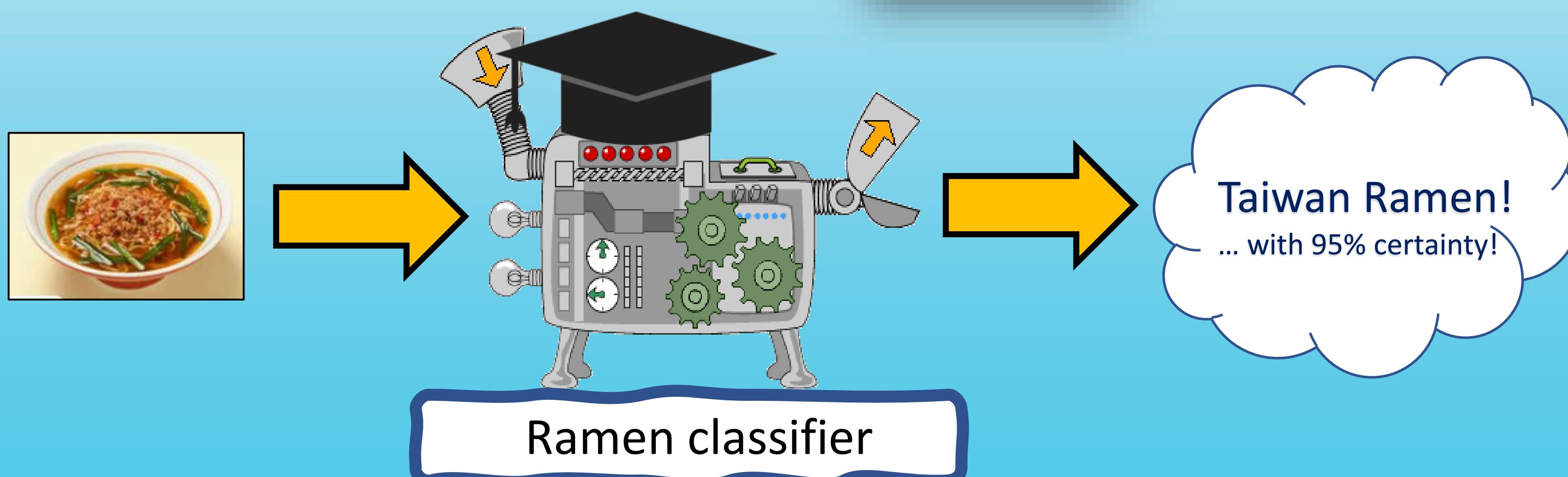
We can simulate "brains" and teach them!



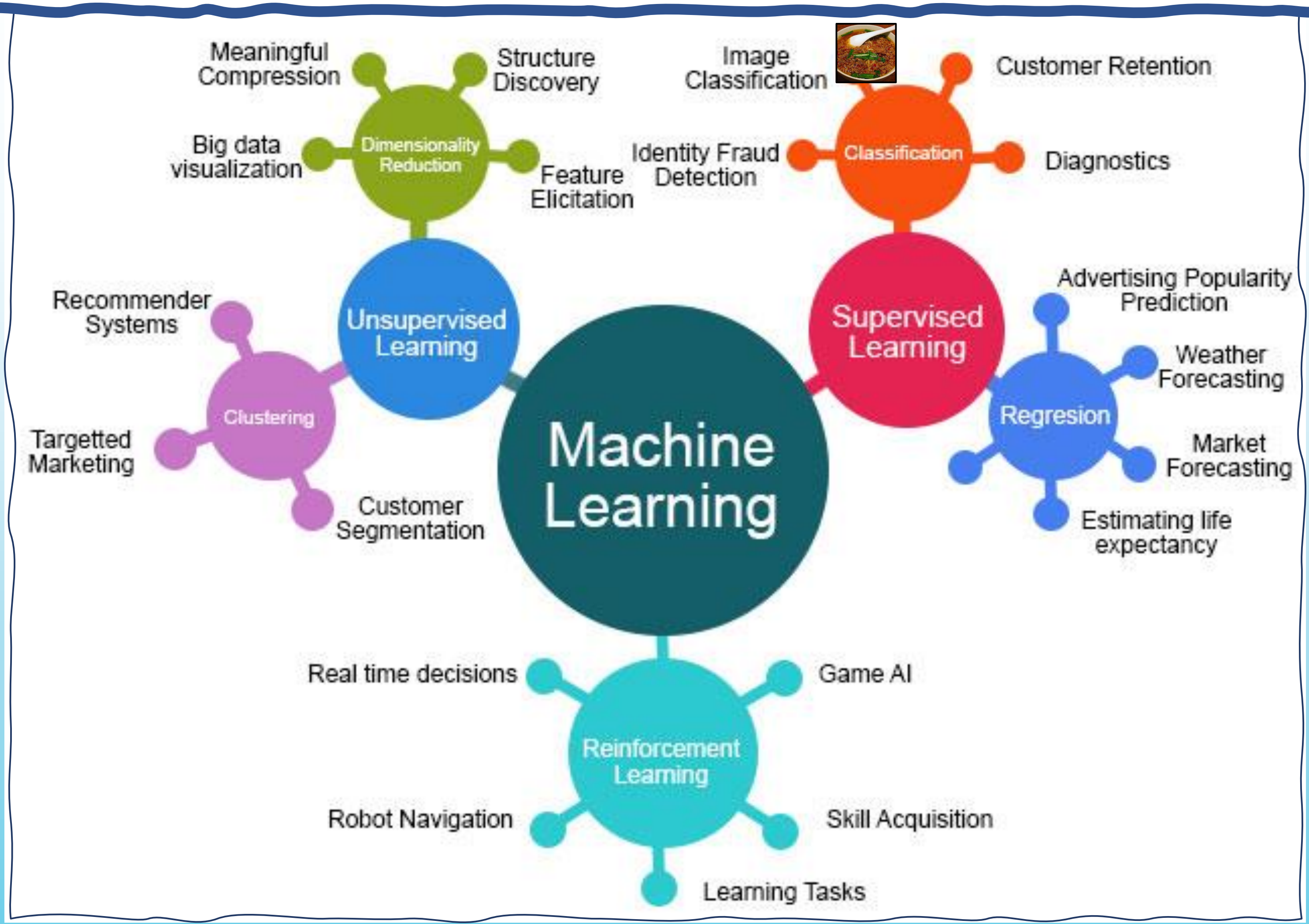




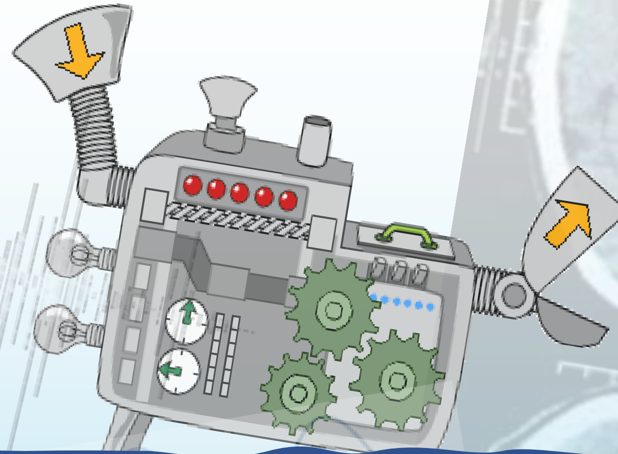
Neural network



Machine learning



In Fall 2020 I am planning to offer a „Math for machine learning" course in the G30 Program.



Thank you very much for your attention!