Formal double zeta space

形式的な二重ゼータ値の空間

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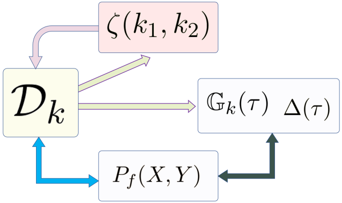

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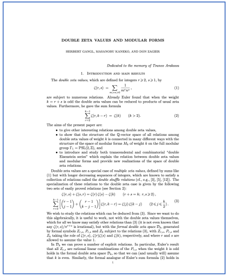
References



DOUBLE ZETA VALUES AND MODULAR FORMS

HERBERT GANGL, MASANOBU KANEKO, AND DON ZAGIER

Dedicated to the memory of Tsuneo Arakawa

1. Introduction and main results

The *double zeta values*, which are defined for integers $r \geqslant 2$, $s \geqslant 1$, by

$$\zeta(r,s) = \sum_{m>n>0} \frac{1}{m^r n^s}, \tag{1}$$

are subject to numerous relations. Already Euler found that when the weight $k = r+s$ is odd the double zeta values can be reduced to products of usual zeta values. Furthermore, he gave the sum formula

$$\sum_{r=2}^{k-1} \zeta(r, k-r) = \zeta(k) \qquad (k>2). \tag{2}$$

The aims of the present paper are:

- to give other interesting relations among double zeta values,
- to show that the structure of the $\mathbb{Q}$-vector space of all relations among double zeta values of weight k is connected in many different ways with the structure of the space of modular forms M_k of weight k on the full modular group $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$, and
- to introduce and study both transcendental and combinatorial "double Eisenstein series" which explain the relation between double zeta values and modular forms and provide new realizations of the space of double zeta relations.

Double zeta values are a special case of *multiple zeta values*, defined by sums like (1) but with longer decreasing sequences of integers, which are known to satisfy a collection of relations called the *double shuffle relations* (cf., e.g., [3], [5], [12]). The specialization of these relations to the double zeta case is given by the following two sets of easily proved relations (see Section 2):

$$\zeta(r,s) + \zeta(s,r) = \zeta(r)\zeta(s) - \zeta(k) \qquad (r+s=k;\ r,s \geqslant 2),$$
$$\sum_{r=2}^{k-1} \left[\binom{r-1}{j-1} + \binom{r-1}{k-j-1}\right] \zeta(r,k-r) = \zeta(j)\zeta(k-j) \qquad (2 \leqslant j \leqslant \tfrac{k}{2}). \tag{3}$$

We wish to study the relations which can be deduced from (3). Since we want to do this algebraically, it is useful to work, not with the double zeta values themselves, which for all we know may satisfy other relations than (3) (it is not even known that any $\zeta(r,s)/\pi^{r+s}$ is irrational), but with the *formal double zeta space* $\mathcal{D}_k$, generated by formal symbols $Z_{r,s}$, $P_{r,s}$ and Z_k subject to the relations (3), with $Z_{r,s}$, $P_{r,s}$ and Z_k taking the role of $\zeta(r,s)$, $\zeta(r)\zeta(s)$ and $\zeta(k)$, respectively, and where r and s are allowed to assume the value 1.

In $\mathcal{D}_k$ we can prove a number of explicit relations. In particular, Euler's result that all $Z_{r,s}$ are rational linear combinations of the $P_{r,s}$ when the weight k is odd holds in the formal double zeta space $\mathcal{D}_k$, so that we can (and usually will) assume that k is even. Similarly, the formal analogue of Euler's sum formula (2) holds in

1

[GKZ]

H. Gangl, M. Kaneko, D. Zagier: "Double zeta values and modular forms"
in "Automorphic forms and zeta functions" World Sci. Publ., Hackensack, NJ (2006), 71-106.

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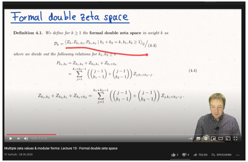
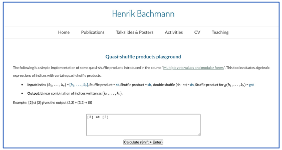
https://www.henrikbachmann.com/mzv2020.html

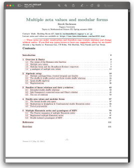
H. Bachmann: Lecture "Multiple zeta values and modular forms"

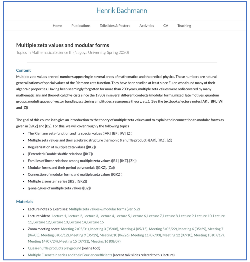
- Lecture notes (Chapter 4)
- Lecture videos
- Interactive online tool for double shuffle relations

[B]









Talk overview
概要と辞書

形式的な二重ゼータ値の空間
Formal double zeta space $\mathcal{D}_k$

$$P_{k_1,k_2} = Z_{k_1,k_2} + Z_{k_2,k_1} + Z_{k_1+k_2}$$
$$= \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) Z_{j,k_1+k_2-j}$$

工具箱
Mathematical toolbox to deal with double shuffle relations

- Generating series　母関数
- Group action　群作用

Sum formula

Parity theorem
偶奇性

$\mathcal{P}_k^{\mathrm{ev}}$

Motivates

Double zeta values　二重ゼータ値
& double shuffle relations
ダブルシャッフル関係式

$$\zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)$$
$$\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)$$

実現
Realizations

Modular forms　モジュラー形式

$$G_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

Eichler-Shimura

Period polynomials　周期多項式

W_k^-
W_k^+

$$P_f(X,Y) = \int_0^{i\infty} (X-Y\tau)^{k-2} f(\tau)\,d\tau$$

Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}\,.$$

There are two different ways to express the product of MZV in terms of MZV.

Stuffle product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j)\,.$$

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\begin{aligned}\zeta(2)\cdot\zeta(3) &\overset{\text{stuffle}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\\ &\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.\end{aligned}$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

Online tool: `https://www.henrikbachmann.com/shuffle.html`.

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0}\frac{1}{m^2 n} = \zeta(2,1) = \zeta(3) = \sum_{m>0}\frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations

$\rightsquigarrow$**extended double shuffle relations**.

In low depth the shuffle (ш) and stuffle ($*$) regularized multiple zeta values are the following elements in $\mathbb{R}[T]$

$$\begin{aligned}
\zeta^{ш}(1;T) &= \zeta^*(1;T) = T \\
\zeta^{ш}(k;T) &= \zeta^*(k;T) = \zeta(k) && (k \geq 2) \\
\zeta^{ш}(k_1,k_2;T) &= \zeta^*(k_1,k_2;T) = \zeta(k_1,k_2) && (k_1 \geq 2, k_1 \geq 1) \\
\zeta^{ш}(1,k_2;T) &= \zeta^*(1,k_2;T) = T\zeta(k_2) - \zeta(k_2,1) - \zeta(k_2+1) && (k_2 \geq 2)
\end{aligned}$$

The only case where $\zeta^{ш}$ and ζ^* differ in depth two is in weight 2:

$$\zeta^{ш}(1,1;T) = \zeta^*(1,1;T) + \frac{1}{2}\zeta(2) = \frac{1}{2}T^2 .$$

Using the stuffle and shuffle regularized multiple zeta values, we have for all $k_1, k_2 \geq 1$

$$\begin{aligned}
\zeta^*(k_1;T)\zeta^*(k_2;T) &= \zeta^*(k_1,k_2;T) + \zeta^*(k_2,k_1;T) + \zeta^*(k_1+k_2;T) \\
&= \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta^{ш}(j, k_1+k_2-j;T) .
\end{aligned}$$

(See: K. Ihara, M. Kaneko, and D. Zagier: *Derivation and double shuffle relations for multiple zeta values*)

Now define for $\bullet \in \{ш, *\}$ their generating series

$$\mathfrak{T}^\bullet(X) = \sum_{k\geq 1} \zeta^\bullet(k;T)X^{k-1}\,, \qquad \mathfrak{T}^\bullet(X,Y) = \sum_{k_1,k_2\geq 1} \zeta^\bullet(k_1,k_2;T)X^{k_1-1}Y^{k_2-1}\,.$$

Using $\frac{X^{k-1}-Y^{k-1}}{X-Y} = \sum_{k_1+k_2=k} X^{k_1-1}Y^{k_2-1}$ we see that

$$\begin{aligned}\mathfrak{T}^\bullet(X)\mathfrak{T}^\bullet(Y) &= \mathfrak{T}^\bullet(X,Y) + \mathfrak{T}^\bullet(Y,X) + \frac{\mathfrak{T}^\bullet(X)-\mathfrak{T}^\bullet(Y)}{X-Y} - \delta_{\bullet,ш}\zeta(2)\\ &= \mathfrak{T}^\bullet(X+Y,Y) + \mathfrak{T}^\bullet(X+Y,X) + \delta_{\bullet,*}\zeta(2)\,,\end{aligned}$$

where δ denotes the Kronecker-delta. This correction term comes from the difference

$$\zeta^{ш}(1,1;T) = \zeta^*(1,1;T) + \frac{1}{2}\zeta(2)\,.$$

Definition

We define for $k \geq 1$ the **formal double zeta space** of weight k as

$$\mathcal{D}_k = \langle Z_k, Z_{k_1,k_2}, P_{k_1,k_2} \mid k_1 + k_2 = k, k_1, k_2 \geq 1 \rangle_{\mathbb{Q}} \Big/ (\heartsuit\mathrm{DS}\heartsuit)$$

where we divide out the following relations for $k_1, k_2 \geq 1$

$$\begin{aligned} P_{k_1,k_2} &= Z_{k_1,k_2} + Z_{k_2,k_1} + Z_{k_1+k_2} \\ &= \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) Z_{j,k_1+k_2-j} \,. \end{aligned} \qquad (\heartsuit\mathrm{DS}\heartsuit)$$

② Formal double zeta space - In small weight

k	Relations in $\mathcal{D}_k$	Basis of $\mathcal{D}_k$	$\dim_{\mathbb{Q}} \mathcal{D}_k$
1	-	Z_1	1
2	$Z_2 = 0, \quad P_{1,1} = 2Z_{1,1}$	$Z_{1,1}$	1
3	$Z_{2,1} = Z_3, \quad Z_{1,2} = P_{1,2} - 2Z_3$	$Z_3, P_{1,3}$	2
4	$Z_4 = 4Z_{3,1}, \quad Z_{2,2} = 3Z_{3,1},$ $P_{1,3} = Z_{1,3} + 5Z_{3,1}, \quad P_{2,2} = 10Z_{3,1}$	$Z_{1,3}, Z_{3,1}$	2
5	$Z_{4,1} = 2Z_5 - P_{2,3}, \quad Z_{3,2} = -\frac{11}{2}Z_5 + 3P_{2,3},$ $Z_{2,3} = \frac{9}{2}Z_5 - 2P_{2,3}, \quad Z_{1,4} = -3Z_5 + P_{1,4} + P_{2,3}$	$Z_5, P_{1,4}, P_{2,3}$	3
6	$Z_6 = 4Z_{3,3} + 4Z_{5,1}, \quad \ldots$	$Z_{1,5}, Z_{3,3}, Z_{5,1}$	3

(Since $P_{k_1,k_2} = P_{k_2,k_1}$ is symmetric we just consider the case $k_1 \leq k_2$.)

- Number of generators: k $\qquad (Z_k, Z_{1,k}, \dots, Z_{k-1,1})$

Since (♡DS♡) symmetric in k_1 and k_2 we obtain for k even $\frac{k}{2}$ relations, i.e.

$$\dim_{\mathbb{Q}} \mathcal{D}_k \geq \frac{k}{2} \qquad (k \text{ even}).$$

For k odd we have $\frac{k-1}{2}$ relations and therefore

$$\dim_{\mathbb{Q}} \mathcal{D}_k \geq \frac{k+1}{2} \qquad (k \text{ odd}).$$

We will see that these are indeed equalities and not just lower bounds.

Let A be a $\mathbb{Q}$-vector space.

Definition

An element in $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{D}_k, A)$ is called a **realization of the double zeta space of weight k in A**.

Remark

We have the following identification

$$\mathrm{Hom}_{\mathbb{Q}}(\mathcal{D}_k, A) \cong \left\{ (\mathsf{Z}_k, \mathsf{Z}_{1,k-1}, \dots, \mathsf{Z}_{k-1,1}) \in A^k \mid \text{satisfying } (\spadesuit) \right\},$$

where

$$\mathsf{Z}_{k_1,k_2} + \mathsf{Z}_{k_2,k_1} + \mathsf{Z}_{k_1+k_2} = \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \mathsf{Z}_{j,k_1+k_2-j}\,. \qquad (\spadesuit)$$

In other words: A realization is just a particular choice of $\mathsf{Z}_k, \mathsf{Z}_{k_1,k_2} \in A$ which satisfy ($\spadesuit$) for all $k_1 + k_2 = k$.

Let A be a $\mathbb{Q}$-vector space.

Definition

An element in $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{D}_k, A)$ is called a **realization of the double zeta space of weight k in A.**

Example (Double zeta value realization)

For $A = \mathbb{R}[T]$ a realization of $\mathcal{D}_k$ with $k \geq 1$ is given for $k_1, k_2 \geq 1$ and $k_1 + k_2 = k$ by

$$Z_k \longmapsto \begin{cases} \zeta^{ш}(k;T) & k \neq 2 \\ 0 & k = 2 \end{cases},$$

$$Z_{k_1,k_2} \longmapsto \zeta^{ш}(k_1, k_2; T),$$

$$P_{k_1,k_2} \longmapsto \zeta^{ш}(k_1;T)\zeta^{ш}(k_2;T).$$

Proposition

For $k \geq 3$ there exist realizations in

- $A = \mathbb{Q}$ with $Z_k \longmapsto -\frac{B_k}{2k!}$.
- $A = \mathbb{Q}[[q]]$ with

$$Z_k \longmapsto G_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$$P_{k_1,k_2} \longmapsto G_{k_1} G_{k_2} + \frac{\delta_{k_1,2}}{2k_2} G'_{k_2} + \frac{\delta_{k_2,2}}{2k_1} G'_{k_1}\,,$$

  where $G'_k = q\frac{d}{dq} G_k$.
- $A = \mathcal{O}(\mathbb{H})$ with $Z_k \longmapsto \mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ ($\tau \in \mathbb{H}, q = e^{2\pi i \tau}$)

The images of Z_{k_1,k_2} can be written down explicitly $\rightsquigarrow$ (combinatorial) double Eisenstein series.

(See the bonus slides for details)

It is convenient to consider generating series when working with the formal double zeta space:

$$\mathfrak{Z}_k(X,Y) = \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 1}} Z_{k_1,k_2} X^{k_1-1} Y^{k_2-1}\,,$$

$\mathfrak{Z}$: "double zetas"

$$\mathfrak{P}_k(X,Y) = \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 1}} P_{k_1,k_2} X^{k_1-1} Y^{k_2-1}\,,$$

$\mathfrak{P}$: "products"

$$\mathfrak{R}_k(X,Y) = Z_k \frac{X^{k-1}-Y^{k-1}}{X-Y}\,.$$

$\mathfrak{R}$: "single zetas"

With this the double shuffle relations (♡DS♡) can be written as

$$\begin{aligned}\mathfrak{P}_k(X,Y) &= \mathfrak{Z}_k(X,Y) + \mathfrak{Z}_k(Y,X) + \mathfrak{R}_k(X,Y)\\ &= \mathfrak{Z}_k(X+Y,Y) + \mathfrak{Z}_k(X+Y,X)\,.\end{aligned} \qquad (♡\text{DSgen}♡)$$

Theorem ([GKZ, Theorem 1])

- For all $k \geq 2$ we have

$$\sum_{j=2}^{k-1} Z_{j,k-j} = Z_k \, .$$

- For $k \geq 2$ even, we have

$$\sum_{\substack{j=2 \\ j \text{ even}}}^{k-2} Z_{j,k-j} = \frac{3}{4} Z_k \, , \qquad \sum_{\substack{j=2 \\ j \text{ odd}}}^{k-1} Z_{j,k-j} = \frac{1}{4} Z_k \, .$$

Proof: By (♡DSgen♡) we have

$$D(X,Y) := \mathfrak{Z}_k(X+Y,Y) + \mathfrak{Z}_k(X+Y,X) - \mathfrak{Z}_k(X,Y) - \mathfrak{Z}_k(Y,X) - \mathfrak{R}_k(X,Y) = 0\,.$$

The first statement now follows by taking the case $(X,Y) = (1,0)$, since

$$0 = D(1,0) = \mathfrak{Z}_k(1,0) + \mathfrak{Z}_k(1,1) - \mathfrak{Z}_k(1,0) - \mathfrak{Z}_k(0,1) - Z_k = \sum_{j=1}^{k-1} Z_{j,k-j} - Z_{1,k-1} - Z_k\,.$$

Proof: By (♡DSgen♡) we have

$$D(X,Y) := \mathfrak{Z}_k(X+Y,Y) + \mathfrak{Z}_k(X+Y,X) - \mathfrak{Z}_k(X,Y) - \mathfrak{Z}_k(Y,X) - \mathfrak{R}_k(X,Y) = 0\,.$$

The first statement now follows by taking the case $(X,Y) = (1,0)$, since

$$0 = D(1,0) = \mathfrak{Z}_k(1,0) + \mathfrak{Z}_k(1,1) - \mathfrak{Z}_k(1,0) - \mathfrak{Z}_k(0,1) - Z_k = \sum_{j=1}^{k-1} Z_{j,k-j} - Z_{1,k-1} - Z_k\,.$$

For the second statement first consider for even k

$$0 = D(1,-1) = \mathfrak{Z}_k(0,-1) + \mathfrak{Z}_k(0,1) - \mathfrak{Z}_k(1,-1) - \mathfrak{Z}_k(-1,1) - Z_k = 2\sum_{j=2}^{k-1} (-1)^j Z_{j,k-j} - Z_k\,.$$

Taking $0 = D(1,0) \pm \frac{1}{2} D(1,-1)$ we therefore obtain

$$0 = 2\sum_{\substack{j=2\\ j \text{ even}}}^{k-2} Z_{j,k-j} - \frac{3}{2} Z_k \qquad \text{and} \qquad 0 = 2\sum_{\substack{j=2\\ j \text{ odd}}}^{k-1} Z_{j,k-j} - \frac{1}{2} Z_k\,.$$

- $V_k \subset \mathbb{Q}[X,Y]$: homogeneous polynomials of degree $k-2$.
- The generating polynomials $\mathfrak{Z}_k, \mathfrak{P}_k, \mathfrak{R}_k$ are elements in $\mathcal{D}_k \otimes_{\mathbb{Q}} V_k$.

On V_k we define a right-action of $\mathrm{Gl}_2(\mathbb{Z})$ for a $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{Z})$ and $F \in V_k$ by

$$(F|\gamma)(X,Y) = F(aX+bY, cX+dY)\,.$$

We extend this action linearly to an action of the group ring $\mathbb{Z}[\mathrm{Gl}_2(\mathbb{Z})]$ on $\mathcal{D}_k \otimes_{\mathbb{Q}} V_k$.

Example We have

$$(F|1+\epsilon)(X,Y) = F(X,Y) + F(Y,X)\,, \qquad (F|T(1+\epsilon))(X,Y) = F(X+Y,Y) + F(X+Y,X)\,,$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

The following elements in $\mathrm{Gl}_2(\mathbb{Z})$ will be of importance when working with the above group action.

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

As seen in the example we can rewrite (♡DSgen♡) as

$$\begin{aligned} \mathfrak{P}_k &= \mathfrak{Z}_k \,|\, (1+\epsilon) + \mathfrak{R}_k \\ &= \mathfrak{Z}_k \,|\, T(1+\epsilon)\,. \end{aligned} \tag{♡DSaction♡}$$

As a consequence of (♡DSaction♡) we can prove the following useful lemma.

Lemma ([B, Lemma 4.5])

For $k \geq 1$ and $A = \epsilon U \epsilon$ we have

$$\mathfrak{z}_k \mid (1-\sigma) = \mathfrak{P}_k \,|(1-\delta)(1+A-SA^2) - \mathfrak{R}_k \,|(1+A+A^2)\,.$$

Since

$$(\mathfrak{P}_k \,|\, \delta)(X,Y) = \mathfrak{P}_k(-X,Y)\,,$$
$$(\mathfrak{z}_k \mid \sigma)(X,Y) = \mathfrak{z}_k(-X,-Y) = (-1)^k \mathfrak{z}_k(X,Y)$$

we have

$$\mathfrak{z}_k \mid (1-\sigma) = \begin{cases} 2\,\mathfrak{z}_k & ,\, k \text{ odd} \qquad \text{(the lemma gives expression of } Z_{k_1,k_2} \text{ in terms of } P_{\text{ev,od}} \text{ and } Z_k\text{)} \\ 0 & ,\, k \text{ even} \qquad \text{(the lemma gives relations among } P_{\text{ev,ev}} \text{ and } Z_k\text{)} \end{cases}.$$

Notice that $A = \epsilon U \epsilon = T\epsilon T^{-1}\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ and that we have $A^3 = \sigma$. By (♡DSaction♡) we get

$$\mathfrak{Z}_k \,|\epsilon = -\mathfrak{Z}_k + \mathfrak{P}_k - \mathfrak{R}_k \quad \text{and} \quad \mathfrak{Z}_k \,|T\epsilon T^{-1} = -\mathfrak{Z}_k + \mathfrak{P}_k \,|T^{-1}\,.$$

Therefore

$$\mathfrak{Z}_k \,|A = \mathfrak{Z}_k \,|(T\epsilon T^{-1})\epsilon = \left(-\mathfrak{Z}_k + \mathfrak{P}_k \,|T^{-1}\right) |\epsilon = \mathfrak{Z}_k + \underbrace{\mathfrak{P}_k \,|(T^{-1}\epsilon - 1) + \mathfrak{R}_k}_{=:\mathfrak{K}}\,.$$

Iterating this identity two more times gives

$$\mathfrak{Z}_k \,|A^3 = \mathfrak{Z}_k + \mathfrak{K} \,|(1 + A + A^2)\,.$$

By direct calculation one can check that the action of $(T^{-1}\epsilon - 1)(1 + A + A^2)$ and $-(1-\delta)(1 + A - SA^2)$ is the same on the symmetric (i.e. ϵ invariant) polynomial $\mathfrak{P}_k$.

$(\epsilon SA^2 = T^{-1}\epsilon,\ \delta = T^{-1}\epsilon A,\ \delta A = T^{-1}\epsilon A^2,\ \epsilon\delta SA = A^2)$

Theorem (Parity)

For odd $k \geq 3$, every Z_{k_1,k_2} with $k_1, k_2 \geq 1$ and $k_1 + k_2 = k$ can be written as a linear combination of $P_{\text{ev,od}}$ and Z_k. More precisely we have

$$Z_{k_1,k_2} = (-1)^{k_2} \sum_{\substack{j=2 \\ j \text{ even}}}^{k-1} \left(\binom{k-j-1}{k_1-1} + \binom{k-j-1}{k_2-1} + \delta_{j,k_1} \right) P_{j,k-j} + \frac{1}{2} \left((-1)^{k_1} \binom{k_1+k_2}{k_2} - 1 \right) Z_k .$$

Example

$$Z_{1,2} = P_{2,1} - 2Z_3 \,, \qquad Z_{2,3} = -2P_{2,3} + \frac{9}{2} Z_5 \,,$$

$$Z_{2,1} = Z_3 \,, \qquad Z_{3,2} = 3P_{2,3} - \frac{11}{2} Z_5 \,,$$

$$Z_{1,4} = P_{2,3} + P_{4,1} - 3Z_5 \,, \qquad Z_{4,1} = -P_{2,3} + 2Z_5 \,.$$

Theorem ([GKZ], [B, Theorem 4.8])

For odd $k \geq 1$ we have $\dim_{\mathbb{Q}} \mathcal{D}_k = \frac{k+1}{2}$ and the sets

$$B_1 = \{Z_k, P_{2,k-3}, P_{4,k-4}, \ldots, P_{k-1,1}\}, \qquad B_2 = \{Z_k, Z_{1,k-1}, Z_{3,k-3}, \ldots, Z_{k-2,2}\},$$

are both bases of $\mathcal{D}_k$.

- The first set is a basis because of the Parity Theorem.
- One can show that one can invert the formula in the Parity Theorem to write any P_{k_1,k_2} in terms of the basis B_2.

From now on (for the rest of the talk) we assume that k is even.

Theorem ([B, Theorem 4.9])

For all $k_1, k_2 \geq 1$ with $k = k_1 + k_2$ even we have

$$\frac{1}{2}\left(\binom{k_1+k_2}{k_2} - (-1)^{k_1}\right) Z_k = \sum_{\substack{j=2 \\ j\,\text{even}}}^{k-2} \left(\binom{k-j-1}{k_1-1} + \binom{k-j-1}{k_2-1} - \delta_{j,k_1}\right) P_{j,k-j}\,.$$

As a consequence of this theorem we get the following relations

$$6P_{2,6} + 3P_{4,4} = \frac{27}{2} Z_8\,, \qquad 15P_{2,6} + 3P_{4,4} = \frac{57}{2} Z_8\,.$$

Combining these two relations we obtain $P_{4,4} = \frac{7}{6} Z_8$.

- Using the DZV/Eisenstein realization, this gives another proof of $\zeta(4)^2 = \frac{7}{6}\zeta(8)$ and $G_4^2 = \frac{7}{6} G_8$.
- One can also use this theorem to show $\zeta(2m) \in \mathbb{Q}\zeta(2)^m$.

Corollary

We have

$$\frac{k+1}{2} Z_k = \sum_{\substack{j=2 \\ j \text{ even}}}^{k-2} P_{j,k-j}\,, \qquad (k \geq 4)$$

$$\frac{(k+1)(k-1)(k-6)}{12} Z_k = \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \geq 4 \text{ even}}} (k_1-1)(k_2-1)\, P_{k_1,k_2}\,. \qquad (k \geq 8)$$

Proof: The first fomula is the case $(k_1, k_2) = (1, k-1)$ in the previous theorem. The second formula follows by considering $k-3$-times the $(k_1, k_2) = (k-2, 2)$ case and then subtracting 2-times the $(k_1, k_2) = (k-3, 3)$ case. □

Applying Eisenstein realization $\Longrightarrow$ Every Eisenstein series G_k can be written as a polynomial in G_4 and G_6.

Theorem ([GKZ], [B, Theorem 4.12])

For even $k \geq 2$ we have $\dim_{\mathbb{Q}} \mathcal{D}_k = \frac{k}{2}$ and the set of $Z_{\text{od,od}}$, i.e.

$$\{Z_{1,k-1}, Z_{3,k-3}, \dots, Z_{k-1,1}\},$$

is a basis of $\mathcal{D}_k$.

There are also explicit formulas to write every element in above basis:

Proposition ([GKZ, eq. (7)], [B, Proposition 4.15])

For $m, n \geq 2$ even and $k = m + n$ we have

$$Z_{m,n} = \frac{2}{1-m} \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \geq 1 \text{ odd}}} \left(\sum_{j=0}^{\min\{k_1-2,n\}} \binom{k-2-j}{m-2} \binom{k_1-1}{j} B_{n-j} \right) \left(Z_{k_1,k_2} + \frac{1}{2} Z_k \right) - \frac{1}{2} Z_k \,.$$

Definition

For a cusp for $f \in S_k$ we define its **period polynomial** as the following polynomial in $\mathbb{C} \otimes V_k$

$$P_f(X,Y) = \int_0^{i\infty} (X - Y\tau)^{k-2} f(\tau)\, d\tau\,.$$

(Nice paper: Don Zagier: *Periods of modular forms and Jacobi theta functions*, Invent. math. 104 (1991) 449-465)

Lemma

For a cusp form $f \in S_k$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$(P_f\,|\gamma)(X,Y) = \int_{\gamma^{-1}(0)}^{\gamma^{-1}(i\infty)} (X - Y\tau)^{k-2} f(\tau)\, d\tau\,,$$

where $\gamma(z) = \frac{az+b}{cz+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathbb{C} \cup \{i\infty\}$.

A consequence of the Lemma is the following:

$$
\begin{aligned}
(P_f \,|\, (1+S))(X,Y) &= \left(\int_0^{\infty i} + \int_{\infty i}^{0}\right)(X - Y\tau)^{k-2} f(\tau)\, d\tau = 0\,, \\
(P_f \,|\, (1+U+U^2))(X,Y) &= \left(\int_0^{\infty i} + \int_1^{0} + \int_{\infty i}^{1}\right)(X - Y\tau)^{k-2} f(\tau)\, d\tau = 0\,.
\end{aligned}
\qquad (☺)
$$

Definition

For even $k \geq 2$ we define

$$W_k = \ker(1+S) \cap \ker(1+U+U^2) \subset V_k\,.$$

By (☺) we have $P_f \in \mathbb{C} \otimes W_k$ for any cusp form $f \in S_k$.

- $V_k^\pm$: symmetric ($+$) and antisymmetric ($-$) polynomials in V_k. $F(X,Y) = \pm F(Y,X)$
- V^{ev} even polynomials in V_k. $F(-X,Y) = F(X,Y)$
- V^{od} odd polynomials in V_k. $F(-X,Y) = -F(X,Y)$
- For $\bullet \in \{+, -, \text{ev}, \text{od}\}$ we write $W_k^\bullet = W_k \cap V_k^\bullet$.

Lemma ([GKZ, p. 14 Lemma])

We have $W_k^+ = W_k^{\text{od}}$, $W_k^- = W_k^{\text{ev}}$ and

$$W_k = \ker(1 - T - T'), \qquad W_k^\pm = \ker(1 - T \mp T\epsilon),$$

where $T' = -U^2 S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

We denote for a cusp form $f \in S_k$ by $P_f^\pm$ the even ($-$) and odd ($+$) parts of P_f in $W_k^\pm$.

Example The first non-trivial cusp form appears in weight 12 and is given by the Delta function

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \,.$$

The odd and even parts of the period polynomial of $f = c\Delta \in S_{12}$ for some explicit $c \in \mathbb{C}$ is given by

$$P_f^+(X,Y) = XY(X^2 - Y^2)^2(X^2 - 4Y^2)(4X^2 - Y^2)\,,$$
$$P_f^-(X,Y) = \frac{36}{691}(X^{10} - Y^{10}) - X^2Y^2(X^2 - Y^2)^3\,.$$

With $W_k^{\pm} = \ker(1 - T \mp T\epsilon)$ we can also write these spaces explicitly as

$$W_k^{+} = \{P \in V_k \mid P(X,Y) - P(X+Y,Y) - P(X+Y,X) = 0\} \ ,$$
$$W_k^{-} = \{P \in V_k \mid P(X,Y) - P(X+Y,Y) + P(X+Y,X) = 0\} \ .$$

Theorem (Eichler-Shimura Isomorphism)

The map $p_f^{\pm} : f \mapsto P_f^{\pm}$ induces isomorphisms

$$p_f^{+} : S_k \xrightarrow{\sim} \mathbb{C} \otimes W_k^{+} \ , \qquad p_f^{-} : S_k \xrightarrow{\sim} \mathbb{C} \otimes {W_k^{-}}\big/{\mathbb{Q}(X^{k-2} - Y^{k-2})} \, .$$

Moreover due to a result of Manin we can always find factors $\omega_+^f, \omega_-^f \in \mathbb{C}$, such that $\frac{1}{\omega_\pm^f} P_f^{\pm} \in W_k^{\pm}$.

Definition

Let $\mathcal{P}_k^{\text{ev}} \subset \mathcal{D}_k$ denote the space spanned by all $P_{\text{ev},\text{ev}}$.

$$\begin{aligned}\mathcal{P}_k^{\text{ev}} &= \langle P_{m,n} \mid m,n \geq 2 \text{ even}, m+n=k \rangle_{\mathbb{Q}} \\ &= \mathbb{Q}Z_k + \langle P_{m,n} \mid m,n \geq 4 \text{ even}, m+n=k \rangle_{\mathbb{Q}} \,,\end{aligned}$$

- Applying the Eisenstein realization φ to this space gives $\varphi(\mathcal{P}_k^{\text{ev}}) = M_k$.
- One can show that the Eisenstein realization is actually an isomorphism from $\mathcal{P}_k^{\text{ev}}$ to M_k.
- In fact it was shown by Gangl, Kaneko and Zagier that $\mathcal{P}_k^{\text{ev}}$ is isomorphic to W_k^-.
- The nice thing about their result is that this isomorphism can be made explicit.

For a $p \in W_k^-$ we define the coefficients $\beta^p_{k_1,k_2} \in \mathbb{Q}$ for $k_1, k_2 \geq 1$, $k_1 + k_2 = k$ by

$$\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 1}} \binom{k-2}{k_1-1} \beta^p_{k_1,k_2} X^{k_1-1} Y^{k_2-1} := p(X+Y, Y).$$

Theorem ([GKZ, Theorem 3], [B, Theorem 4.30])

For even $k \geq 4$ the following map is an isomorphism of $\mathbb{Q}$-vector spaces

$$\begin{aligned} W_k^- &\longrightarrow \mathcal{P}_k^{\text{ev}} \\ p &\longmapsto \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \text{ odd}}} \beta^p_{k_1,k_2} Z_{k_1,k_2} \,. \end{aligned}$$

Moreover the image can be written in terms of the generators of $\mathcal{P}_k^{\text{ev}}$ explicitly as

$$\sum_{\substack{k_1+k_2=k \\ k_1,k_2 \text{ odd}}} \beta^p_{k_1,k_2} Z_{k_1,k_2} \equiv \frac{1}{6} \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \text{ even}}} \beta^p_{k_1,k_2} P_{k_1,k_2} \mod \mathbb{Q} Z_k \,.$$

Corollary

For even $k \geq 4$ and any $p \in W_k^-$ we have

$$\sum_{\substack{k_1+k_2=k \\ k_1,k_2 \text{ odd}}} \beta^p_{k_1,k_2} \zeta(k_1,k_2) \in \mathbb{Q}\pi^k \,.$$

Example The normalized even period polynomial of Δ

$$P^-_{c\Delta}(X,Y) = \frac{36}{691}(X^{10} - Y^{10}) - X^2Y^2(X^2 - Y^2)^3$$

gives the famous relation

$$168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3) = \frac{5197}{691}\zeta(12)\,.$$

Theorem ([GKZ, Proposition 5], [B, Theorem 4.34])

For even $k \geq 4$ the following map is an isomorphism of $\mathbb{Q}$-vector spaces

$$\begin{aligned} W_k^+ &\longrightarrow \{\varphi \in \mathrm{Hom}(\mathcal{P}_k^{\text{ev}}, \mathbb{Q}) \mid \varphi(Z_k) = 0\} \\ \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 2 \text{ even}}} p_{k_1,k_2} X^{k_1-1} Y^{k_2-1} &\longmapsto (\varphi : P_{k_1,k_2} \mapsto p_{k_1,k_2}) \,. \end{aligned}$$

- The above theorem can be interpreted as giving for each odd period polynomial a realization in $\mathbb{Q}$.
- Using the notion of extended period polynomials, the above Theorem can be extended to give realizations in $\mathbb{Q}$ with the images of Z_k being non-zero.
- Applying this general theorem to the period polynomials of Eisenstein series gives the Bernoulli realization (See bonus slides). ([GKZ, Theorem 4 / Supplement to Proposition 5], [B, Theorem 4.40])

Outlook

The tools & ideas for the formal double zeta space might be useful in other contexts.

1. Are there maybe other interesting realizations?
2. There seems to be a connection of $\mathcal{D}_k$ to the space of finite alternating double zeta values.
3. What about a formal finite triple zeta space? Is there a connection to the formal double zeta space and/or period polynomials?($\leftrightsquigarrow$ Kaneko-Zagier conjecture)
4. What about higher depths and/or generalizations? ($\rightsquigarrow$ formal multiple Eisenstein series)
5. Any other ideas?

Thank you for your attention!

Double zeta values 二重ゼータ値

& double shuffle relations

ダブルシャッフル関係式

$$\zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)$$

$$\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)$$

Motivates

形式的な二重ゼータ値の空間

Formal double zeta space $\mathcal{D}_k$

$$P_{k_1,k_2} = Z_{k_1,k_2} + Z_{k_2,k_1} + Z_{k_1+k_2}$$

$$= \sum_{j=1}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right) Z_{j,k_1+k_2-j}$$

工具箱

Mathematical toolbox to deal with double shuffle relations

- Generating series 母関数
- Group action 群作用

Sum formula

Parity theorem

偶奇性

$\mathcal{P}_k^{\mathrm{ev}}$

実現

Realizations

Modular forms モジュラー形式

$$G_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

Eichler-Shimura

Period polynomials 周期多項式

W_k^-

W_k^+

$$P_f(X,Y) = \int_0^{i\infty}(X-Y\tau)^{k-2}f(\tau)\,d\tau$$

Let A be a $\mathbb{Q}$-vector space. (For details see [B] Section 4.2)

Realization lemma ([B, Lemma 4.16])

Assume we have power series $P(X,Y), Z(X,Y), Z(X) \in A[[X,Y]]$ such that

$$\begin{aligned} P(X,Y) &= Z(X,Y) + Z(Y,X) + \frac{Z(X)-Z(Y)}{X-Y} - z(2) \\ &= Z(X+Y,Y) + Z(X+Y,X)\,, \end{aligned}$$

where $Z(X) = \sum_{k\geq 1} z(k) X^{k-1}$. Then φ defined by

$$\begin{aligned} \varphi(Z_k) &= z(k) - \delta_{k,2} z(2)\,, \\ \varphi(Z_{k_1,k_2}) &= \text{coefficient of } X^{k_1-1}Y^{k_2-1} \text{ in } Z(X,Y)\,, \\ \varphi(P_{k_1,k_2}) &= \text{coefficient of } X^{k_1-1}Y^{k_2-1} \text{ in } P(X,Y) \end{aligned}$$

gives a realization of $\mathcal{D}_k$ in A for all $k = k_1 + k_2$.

Theorem ([GKZ, Supplement to Proposition 5], [B, Theorem 4.20])

With $\mathfrak{b}(X) = \sum_{k\geq 1} \beta(k) X^{k-1} = \frac{1}{2}\left(\frac{1}{X} - \frac{1}{e^X - 1} - \frac{1}{2}\right) = \sum_{m\geq 1} \frac{\zeta(2m)}{(2\pi i)^{2m}} X^{2m-1}$ and

$$\mathfrak{b}(X,Y) = \sum_{k_1,k_2\geq 1} \beta(k_1,k_2) X^{k_1-1} Y^{k_2-1}$$
$$:= \frac{1}{3}\big(\mathfrak{b}(X) + \mathfrak{b}(X-Y)\big)\mathfrak{b}(Y) - \frac{5}{12}\frac{\mathfrak{b}(X)-\mathfrak{b}(Y)}{X-Y} + \frac{\mathfrak{b}(X)-\mathfrak{b}(X-Y)}{4Y} - \frac{\mathfrak{b}(Y)-\mathfrak{b}(Y-X)}{12X} - \frac{1}{96}$$

we have

$$\begin{aligned}\mathfrak{b}(X)\mathfrak{b}(Y) &= \mathfrak{b}(X,Y) + \mathfrak{b}(Y,X) + \frac{\mathfrak{b}(X)-\mathfrak{b}(Y)}{X-Y} - \beta(2)\\ &= \mathfrak{b}(X+Y,Y) + \mathfrak{b}(X+Y,X).\end{aligned}$$

In particular this gives a realization φ_β of $\mathcal{D}_k$ in $\mathbb{Q}$ for all k by the Realization lemma.

Definition

Consider the power series

$$\mathfrak{g}(X_1,\dots,X_r) = \sum_{m_1>\cdots>m_r>0} \frac{e^{X_1}q^{m_1}}{1-e^{X_1}q^{m_1}} \cdots \frac{e^{X_r}q^{m_r}}{1-e^{X_r}q^{m_r}}$$

and define the q-series $g(k_1,\dots,k_r) \in \mathbb{Q}[[q]]$ as their coefficients

$$\sum_{k_1,\dots,k_r\geq 1} g(k_1,\dots,k_r) X_1^{k_1-1}\cdots X_r^{k_r-1} := \mathfrak{g}(X_1,\dots,X_r)\,.$$

Proposition

For $k_1 \geq 2, k_2,\dots,k_r \geq 1$ we have

$$\lim_{q\to 1}(1-q)^{k_1+\cdots+k_r} g(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r)\,.$$

Theorem ([GKZ, Theorem 7], [B, Theorem 4.19])

Define the following generating series

$$\mathfrak{h}(X,Y) = \mathfrak{g}(X,Y) - \left(\mathfrak{b}(X-Y) + \frac{1}{2}\right)\mathfrak{g}(X) + \mathfrak{b}(Y)\mathfrak{g}(X) + \mathfrak{b}(X-Y)\mathfrak{g}(Y)$$
$$+ \frac{1}{2}(X-Y)\mathfrak{g}'(Y) + \frac{1}{2}X\mathfrak{g}'(X) + \frac{1}{2}g(2)\,,$$

where $\mathfrak{g}'(X) = q\frac{d}{dq}\sum_{k\geq 1} g(k)\frac{X^{k-1}}{k}$. Then we have

$$\begin{aligned} P(X,Y) &= \mathfrak{h}(X,Y) + \mathfrak{h}(Y,X) + \frac{\mathfrak{g}(X)-\mathfrak{g}(Y)}{X-Y} - g(2) \\ &= \mathfrak{h}(X+Y,Y) + \mathfrak{h}(X+Y,X)\,, \end{aligned}$$

with

$$P(X,Y) = \mathfrak{g}(X)\mathfrak{g}(Y) + \mathfrak{b}(X)\mathfrak{g}(Y) + \mathfrak{b}(Y)\mathfrak{g}(X) + \frac{1}{2}\left(\mathfrak{g}'(X)Y + \mathfrak{g}'(Y)X\right)\,.$$

The theorem on the previous slide gives a realization φ_g of $\mathcal{D}_k$ in $q\mathbb{Q}[[q]]$ for all k by the Realization lemma.

Since the space of realizations $\mathrm{Hom}_{\mathbb{Q}}(\mathcal{D}_k, A)$ is a vector-space, we can add two realizations to get a new one.

Combining the previous two realizations to

$$\varphi_G = \varphi_\beta + \varphi_g$$

gives a realization of $\mathcal{D}_k$ in $\mathbb{Q}[[q]]$ for all k with

$$\varphi_G(Z_k) = G_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\,,$$

$$\varphi_G(P_{k_1,k_2}) = G_{k_1}G_{k_2} + \frac{\delta_{k_1,2}}{2k_2}G'_{k_2} + \frac{\delta_{k_2,2}}{2k_1}G'_{k_1}\,.$$