# q-analogues and finite multiple zeta values


Henrik Bachmann

Nagoya University


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Contact: Math. Building Room 457, henrik.bachmann@math.nagoya-u.ac.jp


Lecture notes are available at: https://www.henrikbachmann.com/qmzv_fmzv.html

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**These notes are under construction and therefore may contain mistakes and change without notice. If you find any typos/errors or have any suggestion, please let me know!**

Parts of these notes are stolen from the lecture notes [B6].

## Contents




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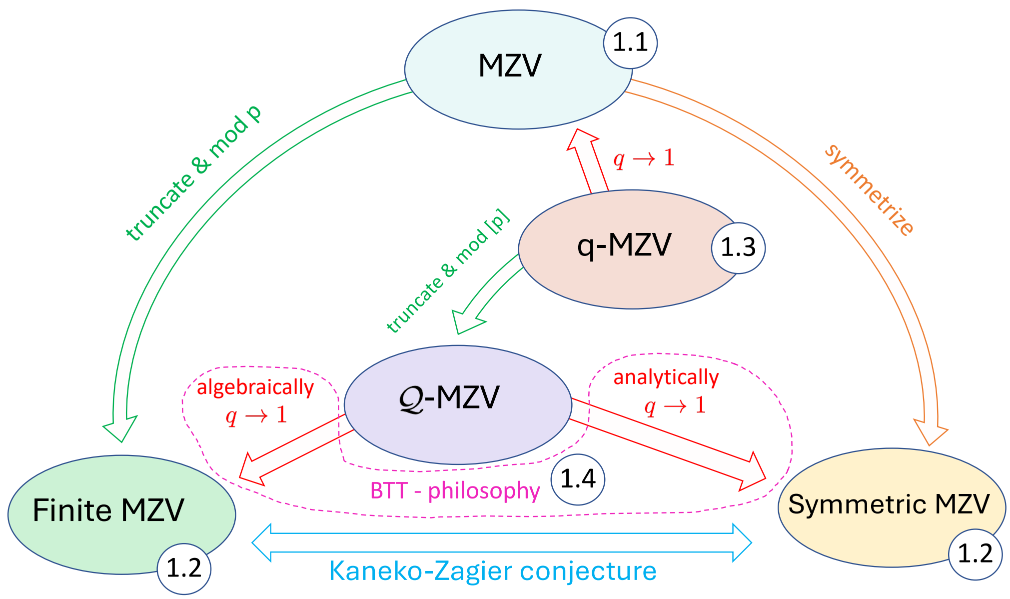
# §1 Overview

In this course, we are interested in the interaction of various variatons of **multiple zeta values (MZV)**, which are for integers $k_1 \geq 2, k_2, \dots, k_r \geq 1$ defined by

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} . \tag{1.1}$$

The goal is to understand the results obtained in [BTT] and parts of [TT]. In particular, we want to understand the following diagram, whose main ingredients will be described briefly in the Subsections 1.1 (MZV), 1.2 (Finite & symmetric MZV), 1.3 ($q$-analogues of MZV), and 1.4 (The connection of $q$-analogues to FMZV and SMZV). After we introduce some algebraic setup in Section 2, we will explain each object in more detail in the Sections 3-5.




The first variation of (1.1) will be given by finite multiple zeta values, which are obtained by truncating the summation to a summation over $p > m_1 > \cdots > m_r > 0$ for a fixed prime $p$. Since all $m_j$ are then smaller than $p$ we can consider the inverse of $m_j$ modulo $p$, i.e. we can consider the sum $\sum_{p>m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \mod p$ as an element in $\mathbb{Z}/p\mathbb{Z}$. Collecting all these over all primes $p$ will give the definition of finite multiple zeta values in Subsection 1.2. Considering their analogues for the 'prime at infinity' will then lead to symmetric multiple zeta values. These can also be seen as a symmetrized version of (regularized) multiple zeta values. Even though finite and symmetric multiple zeta values live in completely different spaces, there is a conjecture, the Kaneko-Zagier conjecture, which claims a surprising isomorphism between them. The $q$-analogues of multiple zeta values will be another variation of multiple zeta values. The basic idea is to replace in (1.1) the $m_i$ in the denominator by their $q$-version $[m]_q = \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1}$ (plus some variation in the numerator). Since $\lim_{q\to1}[m]_q = m$, we will obtain a $q$-series in $\mathbb{Q}[[q]]$, which evalutes to (1.1) as $q \to 1$. Finally, we will combine all the above mentioned objects by considering $q$-analogues at roots of unity. In Subsection 1.4

we will illustrate the main idea, which we call BTT-philosophy, which shows that finite and symmetric multiple zeta values can be obtained from $q$-analogues at roots of unity due to an 'algebraic' and 'analytical' limit '$q \to 1$' respectively. Later we will make this more systematic by studying $\mathcal{Q}$-MZV, which were introduced recently by Takeyama and Tasaka in [TT].

## 1.1 Multiple zeta values

The multiple zeta values (1.1) can be seen as the values of a 'multiple' version of the **Riemann zeta function**, which is defined for a complex variable $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{m>0} \frac{1}{m^s}\,.$$

This function appears in various fields of mathematics and theoretical physics and it can be studied from various points of views. It plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics. It is well-known that the Riemann zeta function can be analytically continued to the whole complex plane with a simple pole at $s = 1$. Even though $\zeta(s)$ was already considered by L. Euler (1707 – 1783), it was named after B. Riemann (1826 – 1866), who proved its meromorphic continuation and functional equation and established a relation between its zeros and the distribution of prime numbers. Euler considered explicit values of the Riemann zeta function at integer points and the most famous result in this direction is his solution to the Basel problem, which states that we have

$$\zeta(2) = \sum_{m>0} \frac{1}{m^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}\,.$$

In general Euler proved that $\zeta(2m)$ is always a rational multiple of $\pi^{2m}$ and he gave the following explicit formula in terms of Seki-Bernoulli numbers.

**Proposition 1.1** (Euler, 1734). *For all $m \in \mathbb{Z}_{\geq 1}$ we have*

$$\zeta(2m) = -\frac{B_{2m}}{2(2m)!}(2\pi i)^{2m} \in \mathbb{Q}\pi^{2m}\,,$$

*where $B_n$ denotes the $n$-th* **Seki-Bernoulli number**[1] *defined by their exponential generating series*

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}\,. \tag{1.2}$$

*Proof.* There are various ways to prove this fact and we will give the original approach due to Euler. First, consider the Weierstrass product of the sine function

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n\geq 1}\left(1 - \frac{x^2}{n^2}\right)\,. \tag{1.3}$$

For $x \in \mathbb{C}\backslash\mathbb{Z}$ we can take its logarithmic derivative to obtain the partial fraction expansion of the cotangent

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{n\geq 1}\left(\frac{1}{x+n} + \frac{1}{x-n}\right)\,.$$

[1] These numbers are often just called Bernoulli numbers. Due to the authors affiliation to Japan he prefers using a historically more fair/accurate naming.

Expanding the right hand side in a geometric series gives

$$\frac{1}{x}+\sum_{n\geq 1}\left(\frac{1}{x+n}+\frac{1}{x-n}\right)=\frac{1}{x}-\sum_{m=1}^{\infty}2\zeta(2m)x^{2m-1}\,.$$

On the other hand the left hand side can be evaluated as

$$\pi\cot(\pi x)=\pi i\frac{e^{\pi ix}+e^{-\pi ix}}{e^{\pi ix}-e^{-\pi ix}}=\pi i\left(1+\frac{2}{e^{2\pi ix}-1}\right)\overset{(1.2)}{=}\frac{1}{x}+\sum_{m=1}^{\infty}\frac{B_{2m}(2\pi i)^{2m}}{(2m)!}x^{2m-1}\,,$$

where in the last equality we used $B_1=-\frac{1}{2}$. □

The first explicit values for $\zeta(2m)$ are given by the following

$$\zeta(2)=\frac{\pi^2}{6}\,,\quad \zeta(4)=\frac{\pi^4}{90}\,,\quad \zeta(6)=\frac{\pi^6}{945}\,,\quad \zeta(8)=\frac{\pi^8}{9450}\,,\quad \zeta(10)=\frac{\pi^{10}}{93555}\,,\quad \zeta(12)=\frac{691\pi^{12}}{638512875}\,.$$

Euler also showed that the (meromorphic continuation of the) Riemann zeta function at negative integers is for $k\geq 0$ given by

$$\zeta(-k)=(-1)^k\frac{B_{k+1}}{k+1}.$$

In particular, from this one can read off the trivial zeros $\zeta(-2m)=0$ for $m\geq 1$, since $B_k$ for odd $k\geq 3$ vanish. The remaining zeros of $\zeta$ are subject of the famous Riemann hypothesis, which predicts that the non-trivial zeros of $\zeta$ all have real-part $\frac{1}{2}$.
Since $\pi$ is transcendental (Lindemann, 1882), Proposition 1.1 gives the only family of polynomial relations among even zeta values. On the other hand, one does not expect polynomial relations among odd zetas. This is part of the following folklore conjecture.

**Conjecture 1.2.** *The numbers* $1,\pi^2,\zeta(3),\zeta(5),\zeta(7),\dots$ *are algebraically independent over* $\mathbb{Q}$.

So far there is not much known towards this conjecture. For the odd zeta values the following theorem gives an overview over the known facts.

**Theorem 1.3.** *i)* $\zeta(3)$ *is irrational.* (Apéry, 1978)

*ii)* *For* $m\geq 1$ *we have*

$$\dim_{\mathbb{Q}}\langle 1,\zeta(3),...,\zeta(2m+1)\rangle\geq\frac{1}{3}\log(2m+1)\,.$$

*In particular infinitely many of the values* $\zeta(2m+1)$ *are irrational.* (Ball–Rivoal, 2001)

*iii)* *At least one of the values* $\zeta(5),\zeta(7),\zeta(9)$ *and* $\zeta(11)$ *is irrational.* (Zudilin, 2001)

**Proposition 1.4.** *For integers* $k_1\geq 2$, $k_2,\dots,k_r\geq 1$ *the sum* (1.1) *converges.*

*Proof.* It is enough to show the convergence for $k_1=2$ and $k_2=\dots=k_r=1$ for any $r$, since this gives an estimate for the other cases. Using the well-known inequality $\sum_{n=1}^m\frac{1}{n}\leq 1+\log(m)$ we obtain

$$\sum_{m_1>m_2>\dots>m_r>0}\frac{1}{m_1^2m_2\cdots m_r}=\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{m>m_2>\dots>m_r>0}\frac{1}{m_2\cdots m_r}\leq\sum_{m=1}^{\infty}\frac{1}{m^2}(1+\log(m))^{r-1}$$

and since $(1+\log(m))^{r-1}=o(\sqrt{m})$ as $m\to\infty$ for any $r$, the above sum converges. □

The multiple sum (1.1) will give the definition of the multiple zeta values which we will give after introducing the following notation.

**Definition 1.5.** *i) For $r \geq 0$ we call a tuple $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$ of positive integers an* **index***. For $r = 0$ we write $\mathbf{k} = \emptyset$ and refer to it as the empty index.*

*ii) An index $\mathbf{k} = (k_1, \dots, k_r)$ is called* **admissible** *if $k_1 \geq 2$ or $\mathbf{k} = \emptyset$.*

*iii) For an index $\mathbf{k} = (k_1, \dots, k_r)$ we call $\mathrm{wt}(\mathbf{k}) = k_1 + \dots + k_r$ its* **weight** *and $\mathrm{dep}(\mathbf{k}) = r$ its* **depth***. We set $\mathrm{wt}(\emptyset) = \mathrm{dep}(\emptyset) = 0$.*

**Definition 1.6.** *For an admissible index $\mathbf{k} = (k_1, \dots, k_r)$ we define the* **multiple zeta value** *$\zeta(\mathbf{k})$ by*

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}$$

*and $\zeta(\emptyset) = 1$. In the case $r = 1$ (resp. $r = 2$) we refer to these as single (resp. double) zeta values.*

*Remark* 1.7. i) By Proposition 1.4 the $\zeta(\mathbf{k})$ gives for every admissible index $\mathbf{k}$ a real number. Even though the notion of weight and depth for these real numbers might not be well defined (and indeed we will see already in Proposition 1.9 below that this is not the case for the depth), we also say that $\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r)$ has weight $\mathrm{wt}(\mathbf{k}) = k_1 + \dots + k_r$ and depth $\mathrm{dep}(\mathbf{k}) = r$.

ii) For $r = 1$ the multiple zeta values are given by the values of the Riemann zeta function. One can also define multiple zeta functions $\zeta(s_1, \dots, s_r)$ for complex variables $s_1, \dots, s_r \in \mathbb{C}$ and consider their analytic properties similar to the classical case. See for example the thesis of Onozuka [On] for a nice detailed survey or [Zh1].

As an analogue of Euler's result $\zeta(2) = \frac{\pi^2}{6}$ we have the following.

**Proposition 1.8.** *For $n \geq 1$ we have*

$$\zeta(\underbrace{2, \dots, 2}_{n}) = \zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}.$$

*Proof.* Using the Taylor series and the product expansion of sin (1.3), we can write the generating series of $\zeta(\{2\}^n)$ as

$$1 + \sum_{n \geq 1} (-1)^n \zeta(\{2\}^n) x^{2n} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2}\right) = \frac{\sin(\pi x)}{\pi x} = \sum_{n=0} (-1)^n \frac{(\pi x)^{2n}}{(2n+1)!}.$$

□

Multiple zeta values satisfy various $\mathbb{Q}$-linear relations. The first one appears in weight 3 and is originally due to Euler. During the course we will see several ways to prove it and the interested reader can find 32 ways of doing so in [BB].

**Proposition 1.9.** *We have $\zeta(3) = \zeta(2, 1)$.*

*Proof.* The shortest proof known to the author is the following: Consider the following sum

$$S = \sum_{m,n>0} \frac{1}{mn(m+n)} = \sum_{m,n>0} \frac{1}{n^2} \left(\frac{1}{m} - \frac{1}{m+n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{n} \frac{1}{m} = \zeta(3) + \zeta(2,1).$$

This sum can also be evaluated as follows

$$S = \sum_{m,n>0} \left(\frac{1}{n} + \frac{1}{m}\right) \frac{1}{(m+n)^2} = \sum_{m,n>0} \frac{1}{n(m+n)^2} + \sum_{m,n>0} \frac{1}{m(m+n)^2} = 2\zeta(2,1)$$

and therefore the relation $\zeta(3) = \zeta(2,1)$ follows. □

Another way to obtain relations among multiple zeta values is to evaluate the product $\zeta(k_1)\zeta(k_2)$ in two different ways. For $k_1, k_2 \geq 2$ we get

$$\begin{aligned}\zeta(k_1)\zeta(k_2) = \sum_{m_1>0} \frac{1}{m_1^{k_1}} \sum_{m_2>0} \frac{1}{m_2^{k_2}} &= \left(\sum_{m_1>m_2>0} + \sum_{m_2>m_1>0} + \sum_{m_1=m_2>0}\right) \frac{1}{m_1^{k_1} m_2^{k_2}} \\ &= \zeta(k_1,k_2) + \zeta(k_2,k_1) + \zeta(k_1+k_2)\,.\end{aligned} \tag{1.4}$$

which is often called the **stuffle product** (also called **harmonic product**). But we also have the following expression for the product, which is called the **shuffle product**.

**Proposition 1.10.** *For $k_1, k_2 \geq 2$ we have*

$$\zeta(k_1)\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1} \left( \binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1+k_2-j)\,, \tag{1.5}$$

*where we use the usual convention $\binom{n}{k} = 0$ for $n < k$.*

*Proof.* This follows from the following partial fraction decomposition, which can be proven by induction on $k_1, k_2$ ([BF, Lemma 1.49])

$$\frac{1}{m^{k_1} n^{k_2}} = \sum_{j=1}^{k_1+k_2} \left( \frac{\binom{j-1}{k_1-1}}{(m+n)^j n^{k_1+k_2-j}} + \frac{\binom{j-1}{k_2-1}}{(m+n)^j m^{k_1+k_2-j}} \right).$$

The sum in (1.5) can start at $j = 2$, since the binomial coefficients for $j = 1$ vanish in the case $k_1, k_2 \geq 2$. □

Comparing the right hand sides of (1.4) and (1.5) gives for $k_1, k_2 \geq 2$ a linear relation among multiple zeta values, which is an example for a so-called **(finite) double shuffle relation**. We will consider the stuffle/harmonic & shuffle product and the resulting double shuffle relations in detail for arbitrary depth in Section 2.

*Example* 1.11. For $k_1 = 2$, $k_2 = 3$ equations (1.4) and (1.5) give

$$\begin{aligned}\zeta(2)\zeta(3) &= \zeta(2,3) + \zeta(3,2) + \zeta(5)\,,\\ \zeta(2)\zeta(3) &= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)\,,\end{aligned}$$

from which we deduce the linear relation $\zeta(5) = 2\zeta(3,2) + 6\zeta(4,1)$.

Until here in lecture 1 (15th April, 2022)

We will denote the $\mathbb{Q}$-vector space spanned by all multiple zeta values by

$$\mathcal{Z} = \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible} \rangle_{\mathbb{Q}} .$$

For a fixed weight $k \geq 0$ we also define the space of weight $k$ multiple zeta values by

$$\mathcal{Z}_k = \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible}, \mathrm{wt}(\mathbf{k}) = k \rangle_{\mathbb{Q}} .$$

Clearly we have $\mathcal{Z} = \sum_{k\geq 0} \mathcal{Z}_k$. With the same idea as in (1.4), where we showed that $\zeta(k_1)\zeta(k_2)$ is a linear combination of multiple zeta values of weight $k_1 + k_2$, we will see in Section 2 that this is true for arbitrary products of multiple zeta values and we will show the following:

**Proposition 1.12.** *The space $\mathcal{Z}$ is a $\mathbb{Q}$-subalgebra of $\mathbb{R}$ and we have $\mathcal{Z}_{k_1} \cdot \mathcal{Z}_{k_2} \subset \mathcal{Z}_{k_1+k_2}$ for $k_1, k_2 \geq 0$.*

All of the relations we saw so far: $\zeta(3) = \zeta(2,1)$ and the finite double shuffle relations are relations among multiple zeta values of the same weight. Indeed it is expected that there exist no $\mathbb{Q}$-linear relations among multiple zeta values of different weights, which is part of the following conjecture.

**Conjecture 1.13.** *The space $\mathcal{Z}$ is graded by weight, i.e.*

$$\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k .$$

This conjecture is very strong as it implies the transcendence of every multiple zeta value of non-zero weight. One of the main interest in the theory of multiple zeta values is to understand all of their $\mathbb{Q}$-linear relations.

In Section 2, we will introduce **stuffle regularized multiple zeta values** $\zeta^*(\mathbf{k};T) \in \mathcal{Z}[T]$ and **shuffle regularized multiple zeta values** $\zeta^{ш}(\mathbf{k};T) \in \mathcal{Z}[T]$, which are defined for arbitrary indices $\mathbf{k}$. These assign to the non-admissible index $\mathbf{k} = (1)$ the indeterminant $\zeta^*(1;T) = \zeta^{ш}(1;T) = T$ and they satisfy for an admissible index $\mathbf{k}$ that $\zeta^*(\mathbf{k};T) = \zeta^{ш}(\mathbf{k};T) = \zeta(\mathbf{k})$. Regularized multiple zeta values satisfy for arbitrary indices analogues of the stuffle and shuffle product formulas. For example, in smallest depth we have as an analogue of (1.4) and (1.5) for all $k_1, k_2 \geq 1$:

$$\zeta^*(k_1;T)\zeta^*(k_2;T) = \zeta^*(k_1,k_2;T) + \zeta^*(k_2,k_1;T) + \zeta^*(k_1+k_2;T) ,$$
$$\zeta^{ш}(k_1;T)\zeta^{ш}(k_2;T) = \sum_{j=1}^{k_1+k_2-1} \left( \binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta^{ш}(j, k_1+k_2-j;T) .$$

For example, for $k_1 = 1$ and $k_2 = 2$ we get

$$\zeta(2)T = \zeta^*(1,2;T) + \zeta(2,1) + \zeta(3),$$
$$\zeta(2)T = \zeta^{ш}(1,2;T) + 2\zeta(2,1) ,$$

from which we get $\zeta^*(1,2;T) = \zeta(2)T - \zeta(3)$ and $\zeta^{ш}(1,2;T) = \zeta(2)T - 2\zeta(2,1)$. Since we already proved that $\zeta(2,1) = \zeta(3)$ we see that $\zeta^*(1,2;T) = \zeta^{ш}(1,2;T)$, but for general indices $\mathbf{k}$ we have $\zeta^*(\mathbf{k};T) \neq \zeta^{ш}(\mathbf{k};T)$. There exists an explicit $\mathbb{R}$-linear map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$, such that $\zeta^{ш}(\mathbf{k};T) = \rho(\zeta^*(\mathbf{k};T))$. In particular we will see that $\rho(1) = 1$ and $\rho(T) = T$, which conversely shows that $\zeta(2,1) = \zeta(3)$, without using Proposition 1.9. Comparing $\zeta^*$ and $\zeta^{ш}$ in a general by this $\rho$ gives a large family of relations, which are called **extended double shuffle relations**. Conjecturally these give all relations among multiple zeta values and we will give a precise version of the following conjecture later.

**Conjecture 1.14.** *All relations among multiple zeta values are a consequence of the extended double shuffle relations.*

There also exists a conjecture for the dimension of the spaces $\mathcal{Z}_k$, which was first observed by Zagier based on extensive numerical calculations. To state the conjecture we first introduce the integers $d_k$ given by the following generating series

$$\sum_{k\geq 0} d_k X^k = \frac{1}{1-X^2-X^3}\,,$$

i.e. they are given by $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and the recursion $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$.

**Conjecture 1.15** (Zagier, 1994)**.** *We have* $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$ *for all* $k \geq 0$.

This conjecture shows that multiple zeta values satisfy a lot of linear relations. For example in weight $k = 14$ there are $2^{12} = 4096$ admissible indices (i.e. generators of $\mathcal{Z}_{14}$) and the conjectured dimension is $d_{14} = 21$. In the following we give a table for the number of admissible indices, the conjectured number of linearly independent relations and the numbers $d_k$.

| weight $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # of adm. ind. | 1 | 0 | 1 | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 | 1024 | 2048 | 4096 |
| # of relations $\stackrel{?}{=}$ | 0 | 0 | 0 | 1 | 3 | 6 | 14 | 29 | 60 | 123 | 249 | 503 | 1012 | 2032 | 4075 |
| $\left(\dim_{\mathbb{Q}} \mathcal{Z}_k \stackrel{?}{=}\right) d_k$ | 1 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 7 | 9 | 12 | 16 | 21 |

The Conjecture 1.15 is out of reach at the moment and so far there is no weight $k$, for which we can actually prove that $\dim_{\mathbb{Q}} \mathcal{Z}_k > 1$, since for example it is not even known (but expected) that $\zeta(5)$ and $\zeta(2,3)$ are linearly independent and that $\dim_{\mathbb{Q}} \mathcal{Z}_5 = 2$. Even though it seems to be impossible to give lower bounds for $\dim_{\mathbb{Q}} \mathcal{Z}_k$ so far, we know that the $d_k$ give upper bounds:

**Theorem 1.16** (Terasoma (2002), Deligne–Goncharov (2005))**.** *For all* $k \geq 0$ *we have* $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.

There also is a conjecture on an explicit basis for $\mathcal{Z}$ due to Hoffman.

**Conjecture 1.17** (Hoffman [H1], 1997)**.** *For* $k \geq 0$ *the multiple zeta values*

$$\{\zeta(k_1,\dots,k_r) \mid r \geq 0,\, k_1+\dots+k_r = k,\, k_1,\dots,k_r \in \{2,3\}\}$$

*form a basis of* $\mathcal{Z}_k$.

Notice that this conjecture would imply Zagiers dimension conjecture (Conjecture 1.15) since $d_k$ counts exactly the number of indices of weight $k$ with only 2's and 3's. Multiple zeta values with only 2's and 3's in their index will be called **Hoffman elements**. The linear independence of the Hoffman elements is unknown so far, but we know that these generate the whole space due to the following deep result of Brown.

**Theorem 1.18** (Brown [Br], 2012)**.** *For all* $k \geq 0$ *we have*

$$\mathcal{Z}_k = \langle \zeta(k_1,\dots,k_r) \mid r \geq 0,\, k_1+\dots+k_r = k,\, k_1,\dots,k_r \in \{2,3\}\rangle_{\mathbb{Q}}\,.$$

The only known proofs of Theorem 1.16 and 1.18 use deep concepts from algebraic geometry, particularly the theory of mixed Tate motives.

*Remark* 1.19. In his work [Br], Brown shows that all the above conjectures hold for so-called motivic multiple zeta values $\mathcal{Z}^{\mathfrak{m}}$, which are conjecturally isomorphic as a $\mathbb{Q}$-algebra to $\mathcal{Z}$. Using the surjective period map per : $\mathcal{Z}^{\mathfrak{m}} \twoheadrightarrow \mathcal{Z}$ Theorem 1.18 is then just a consequence of his more general results. For details on this, we refer to the excellent book [BF].

## 1.2 Finite and symmetric multiple zeta values

The finite multiple zeta values will be variant of multiple zeta values, which are not be given as elements of the real numbers. Instead they will be elements of the following ring $\mathcal{A}$ given by

$$\mathcal{A} = \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \Big/ \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \,.$$

This ring was introduced by Zagier and he refers to it as the "poor man's adeles", due to their similarity (but in comparison simpler definition) to the usual (finite) adeles. Elements in $\mathcal{A}$ are infinite tuples $a = (a_p)_{p \text{ prime}} \in \mathcal{A}$ with $a_p \in \mathbb{Z}/p\mathbb{Z}$, such that for $a = (a_p)_p, b = (b_p)_p \in \mathcal{A}$ we have $a = b$ if and only if $a_p = b_p$ for $p \gg 0$. In other words, two such elements are the same if $a_p \neq b_p$ for just finitely many primes $p$. Due to this we have an embedding $\iota : \mathbb{Q} \to \mathcal{A}$, since for $\frac{a}{b} \in \mathbb{Q}$ we can get a solution $x_p$ of

$$b\, x_p - a \equiv 0 \mod p$$

for all but finitely many $p$ (those, where $p|b$). Choose $x_p$ arbitrary if it does not exists and define

$$\iota\left(\frac{a}{b}\right) = (x_2, x_3, x_5, x_7, \dots) \in \mathcal{A}\,.$$

For example, we have

$$\iota\left(\frac{7}{15}\right) = (1, *_3, *_5, 0, 10, 10, 5, 3, 2, 14, 17, 35, 36, 32, 13, \dots)\,,$$

where the $*_p$ can be any element in $\mathbb{Z}/p\mathbb{Z}$. Notice that $\iota$ is injective and therefore $\mathcal{A}$ becomes a $\mathbb{Q}$-algebra. The finite multiple zeta value will be given by a collection of rational numbers depending on prime $p$. For this we first define the truncated version of multiple zeta values, called **multiple harmonic sums**, for $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and $m \geq 1$ by

$$H_m(\mathbf{k}) = H_m(k_1, \dots, k_r) = \sum_{m \geq m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{Q}\,. \tag{1.6}$$

The $H_m(1)$ are the classical harmonic numbers. Notice that for a prime $p$, the element $H_{p-1}(k_1, \dots, k_r) \mod p \in \mathbb{Z}/p\mathbb{Z}$ is well-defined, since all $m_j$ in the numerator are smaller than $p$. Collecting these for all prime $p$ gives our main object for this section:

**Definition 1.20.** *For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ the* **finite multiple zeta value** *is defined by*

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \zeta_{\mathcal{A}}(k_1, \dots, k_r) = (H_{p-1}(\mathbf{k}) \mod p)_p = \left( \sum_{p > m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_p \in \mathcal{A}\,.$$

*Here we also set $\zeta_{\mathcal{A}}(\emptyset) = 1$.*

Notice that we also allowed negative $k_j$, since all the appearing sums are finite and there are no convergence issues. But as we will see in Exercise 4, one can express the finite multiple zeta values with negative entries always in terms of those with positive entries. Therefore, we define the space of finite multiple zeta values as the following $\mathbb{Q}$-vector space

$$\mathcal{Z}^{\mathcal{A}} = \langle \zeta_{\mathcal{A}}(k_1, \dots, k_r) \mid r \geq 0,\, k_1, \dots, k_r \geq 1 \rangle_{\mathbb{Q}} = \langle \zeta_{\mathcal{A}}(\mathbf{k}) \mid \mathbf{k} \text{ index} \rangle_{\mathbb{Q}}\,.$$

For a fixed weight $k \geq 0$ we also define the space of finite multiple zeta values of weight $k$ as

$$\mathcal{Z}^{\mathcal{A}}_k = \langle \zeta_{\mathcal{A}}(\mathbf{k}) \mid \mathbf{k} \text{ index}, \mathrm{wt}(\mathbf{k}) = k \rangle_{\mathbb{Q}}\,.$$

Notice that $\mathbb{Q} \subset \mathcal{Z}^{\mathcal{A}}$ since we also include the empty index ($r = 0$).

**Proposition 1.21** (Exercise 4)**.** *For any $k_1,\dots,k_r \in \mathbb{Z}$ we have $\zeta_{\mathcal{A}}(k_1,\dots,k_r) \in \mathcal{Z}^{\mathcal{A}}$.*

In the following we will consider finite multiple zeta values in small depths. For this we will recall the following two statements, which will be used several times when doing calculations with finite multiple zeta values.

**Lemma 1.22.** *(i)* (Fermat's little theorem) *For a prime $p$ and $m \in \mathbb{Z}$ we have*

$$m^p \equiv m \mod p\,.$$

*(ii)* (Seki-Bernoulli formula) *For $k \geq 1$ we have*

$$\sum_{m=1}^{n-1} m^k = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j} B_j n^{k+1-j}\,.$$

*Proof.* There are various simple proofs for (i). For example, one can reduce the problem to $m \geq 0$ and prove it by induction on $m$. For the induction step one proves the 'Freshman's dream' $(a+b)^p \equiv a^p + b^p \mod p$ for $a,b \in \mathbb{Z}$, which can be shown by using that $\binom{p}{j} \equiv 0 \mod p$ for $0 < j < p$.
For (ii) one can consider the exponential generating series of the left-hand side and gets

$$\sum_{k=1}^{\infty}\left(\sum_{m=1}^{n-1} m^k\right)\frac{X^{k-1}}{(k-1)!} = \sum_{m=1}^{n-1} m e^{mX} = \frac{\partial}{\partial X}\frac{1-e^{nX}}{1-e^X} = \frac{\partial}{\partial X}\frac{X}{e^X-1}\frac{e^{nX}-1}{X}$$
$$= \frac{\partial}{\partial X}\left(\sum_{j=0}^{\infty} B_j \frac{X^j}{j!}\right)\left(\sum_{m=1}^{\infty} n^m \frac{X^{m-1}}{m!}\right) = \sum_{k=1}^{\infty}\left(k\sum_{j=0}^{k}\frac{B_j}{j!}\frac{n^{k+1-j}}{(k+1-j)!}\right)X^{k-1}\,.$$

Comparing the coefficient of $X^{k-1}$ then yields the result. □

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In particular, if $m$ is coprime to $p$ (e.g. $p > m > 0$), then $m^{p-1} \equiv 1 \mod p$. Combining (i) and (ii) in Lemma 1.22 yields for $n \geq 1$, $k \in \mathbb{Z}$ with $k \neq 0$ and primes $p > \max(k, n-1)$

$$H_{n-1}(k) = \sum_{m=1}^{n-1}\frac{1}{m^k} \equiv \sum_{m=1}^{n-1} m^{p-k-1} \equiv -\frac{1}{k}\sum_{j=0}^{p-k-1}\binom{p-k}{j} B_j n^{p-k-j} \mod p. \tag{1.7}$$

The equation (1.7) can be used to evaluate finite multiple zeta values in depth one and two.

**Proposition 1.23.** *For $k \in \mathbb{Z}$ we have*

$$\zeta_{\mathcal{A}}(k) = (H_{p-1}(k) \mod p)_p = \begin{cases} 0, & k \neq 0 \\ -1, & k = 0 \end{cases}.$$

*Proof.* The $k=0$ case is clear and the other cases follow from (1.7) by using $p = n$. Alternatively one can choose a primitive root $a$ modulo $p$ to get for $k \neq 0$ (without using the Seki-Bernoulli formula)

$$\sum_{m=1}^{p-1}\frac{1}{m^k} \equiv \sum_{i=0}^{p-2}\frac{1}{a^{ik}} \equiv \frac{1-a^{-k(p-1)}}{1-a^{-k}} \equiv 0 \mod p\,.$$

□

Proposition 1.23 shows that finite multiple zeta values behave quite differently to multiple zeta values. But, as we will see later, there seems to be a conjectural deep connection between these two. In particular, there exists and element $Z(k) \in \mathcal{Z}^{\mathcal{A}}$ which serves as the correct **analogue of** $\zeta(k)$ instead of the naive choice $\zeta_{\mathcal{A}}(k)$ (in a sense we will make clearer later). For $k \geq 2$ this element is defined by

$$Z(k) = \left(\frac{B_{p-k}}{k}\right)_p \in \mathcal{A}.$$

Notice that this definition makes sense, since we can always ignore small primes $p$, i.e. those cases where the Seki-Bernoulli number $B_{p-k}$ is not defined. Notice that, since $B_{\text{odd}}$ vanish, we have $Z(2m) = 0$ for all $m \geq 1$.

**Proposition 1.24.** *For $k_1, k_2 \geq 1$ we have*

$$\zeta_{\mathcal{A}}(k_1, k_2) = (-1)^{k_1} \binom{k_1 + k_2}{k_1} Z(k_1 + k_2) .$$

*Proof.* For large enough primes $p$ we can use (1.7) to obtain

$$\begin{aligned}\sum_{p > m_1 > m_2 > 0} \frac{1}{m_1^{k_1} m_2^{k_2}} &= \sum_{m_1=1}^{p-1} \frac{1}{m_1^{k_1}} \sum_{m_2=1}^{m_1-1} \frac{1}{m_2^{k_2}} \equiv -\frac{1}{k_2} \sum_{m_1=1}^{p-1} \sum_{j=0}^{p-k_2-1} \binom{p-k_2}{j} B_j m_1^{p-k_2-j-k_1} \\ &\equiv \frac{1}{k_2} \binom{p-k_2}{p-k_1-k_2} B_{p-k_1-k_2} \equiv (-1)^{k_1} \binom{k_1+k_2}{k_1} \frac{B_{p-k_1-k_2}}{k_1+k_2} \mod p ,\end{aligned}$$

where in the third equation we used $\sum_{m_1=1}^{p-1} m_1^{p-k_2-j-k_1} \equiv 0 \mod p$ except for $j = p - k_1 - k_2$. □

Since the odd Seki-Bernoulli number vanish, Proposition 1.24 implies $\mathcal{Z}^{\mathcal{A}}(k_1, k_2) = 0$ if $k_1 + k_2$ is even. For odd weight we expect that these are non-zero, i.e. we have the following.

**Conjecture 1.25.** *For odd $k \geq 3$ we have $Z(k) \neq 0$ in $\mathcal{A}$.*

This conjecture is related to a classical conjecture in algebraic number theory, stating that there are infinitely many regular primes. By a criteria by Kummer a prime $p$ is regular, if (and only if) it does not divide the numerator of the Bernoulli numbers $B_k$ for $k = 2, 4, \ldots, p-3$. If this conjecture would be true we also see that $Z(k) \neq 0$ in $\mathcal{A}$.

As for multiple zeta values we can ask for relations among the finite multiple zeta values and the dimension of the space $\mathcal{Z}_k^{\mathcal{A}}$. For example, as a direct consequence of the definition we have for $k_1, \ldots, k_r \in \mathbb{Z}$ and any prime $p$

$$\begin{aligned}\sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \ldots m_r^{k_r}} &\equiv \sum_{p > m_1 > \cdots > m_r > 0} \frac{(-1)^{k_1 + \cdots + k_r}}{(p - m_1)^{k_1} \ldots (p - m_r)^{k_r}} \\ &\equiv \sum_{0 < \tilde{m}_1 < \cdots < \tilde{m}_r < p} \frac{(-1)^{k_1 + \cdots + k_r}}{\tilde{m}_1^{k_1} \ldots \tilde{m}_r^{k_r}} \mod p ,\end{aligned}$$

which implies the **reversal formula**

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r) = (-1)^{k_1 + \cdots + k_r} \zeta_{\mathcal{A}}(k_r, \ldots, k_1) . \tag{1.8}$$

In the case of multiple zeta values one source of relations (which conjectureally give all) are the double shuffle relations. We have an analogue of the **stuffle product**, e.g. in smallest depth we have for any $k_1, k_2 \geq 1$

$$\zeta_{\mathcal{A}}(k_1)\zeta_{\mathcal{A}}(k_2) = \zeta_{\mathcal{A}}(k_1,k_2) + \zeta_{\mathcal{A}}(k_2,k_1) + \zeta_{\mathcal{A}}(k_1+k_2)\,, \tag{1.9}$$

we do not have that the shuffle product formula (1.5) holds. Instead we will have a family of linear relations, called the **linear shuffle relations**. For example, as an analogue 1.24, we have for $k_1, k_2 \geq 1$

$$(-1)^{k_1}\zeta_{\mathcal{A}}(k_2,k_1) = \sum_{j=1}^{k_1+k_2-1} \left( \binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta_{\mathcal{A}}(j, k_1+k_2-j)\,. \tag{1.10}$$

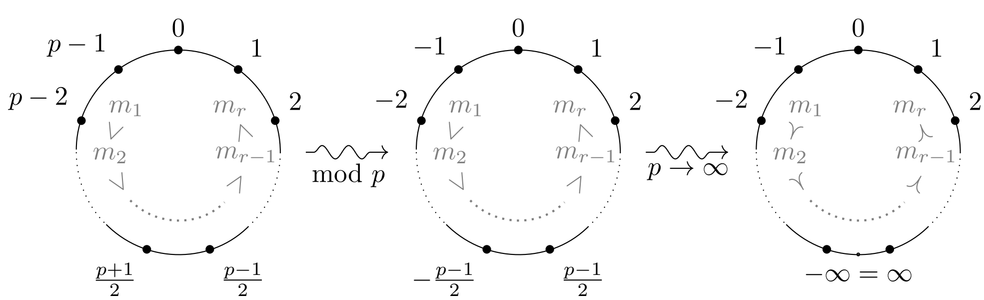
Due to Proposition 1.23 and 1.24 the equations (1.9) and (1.10) do not produce really interesting relations, but for example we have (as a trivial consequence of Proposition 1.24)

$$2\zeta_{\mathcal{A}}(4,1) + \zeta_{\mathcal{A}}(3,2) = 0\,. \tag{1.11}$$

But the higher depths generalization, which we will present in Section 3, actually give interesting and non-trivial relations. Moreover, we have the following conjecture.

**Conjecture 1.26.** *All relations among finite multiple zeta values are a consequence of the stuffle product and the linear shuffle relations.*

The really surprising aspect of the whole story is, that there seems to be a connection to usual multiple zeta values, which we will describe now. For this we define a real counterpart of the finite multiple zeta values introduced by Kaneko-Zagier [KZ]. This can also be seen as finite multiple zeta values at the infinite prime $p \to \infty$. The following idea is due to Kontsevich who communicated it to Zagier for the depth two case. In the definition of finite multiple zeta values we take a sum over $p > m_1 > \cdots > m_r > 0$ and then consider it modulo $p$. Naively one could therefore say that we consider the sum over '$0 > -1 \geq m_1 > \cdots > m_r \geq 1 > 0$', which we illustrate in the following picture.




Inspired by this we consider the **order $\succ$ on** $\mathbb{Z}\backslash\{0\} \cup \{\infty = -\infty\}$ defined by

$$-1 \succ -2 \succ -3 \succ \cdots \succ -\infty = \infty \succ \cdots \succ 3 \succ 2 \succ 1\,.$$

Then we define for and index $\mathbf{k} = (k_1, \ldots, k_r)$ and $m \geq 1$ as an analogue of the multiple harmonic sums the following rational numbers

$$S_m(\mathbf{k}) = S_m(k_1,\ldots,k_r) = \sum_{\substack{m_1 \succ \cdots \succ m_2 \\ m \geq |m_1|,\ldots,|m_r| > 0}} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \in \mathbb{Q}\,. \tag{1.12}$$

By direct calculation one then checks that we have the following.

**Proposition 1.27.** *For all $m \geq 1$ and $k_1, \dots, k_r \geq 1$ we have*

$$S_m(k_1, \dots, k_r) = \sum_{j=0}^{r} (-1)^{k_1+\dots+k_r} H_m(k_j, k_{j-1}, \dots, k_1) H_m(k_{j+1}, \dots, k_r) \,. \tag{1.13}$$

*Here we use $H_m(k_j, k_{j-1}, \dots, k_r) = 1$ (resp. $H_m(k_{j+1}, \dots, k_r) = 1$) for $j = 0$ (resp. $j = r$).*

*Proof.* This is Exercise 5. □

The limit of $H_m(\mathbf{k})$ for $m \to \infty$ just exists if $\mathbf{k}$ is admissible. A priory it is therefore not clear that the limit of (1.13) for $m \to \infty$ exists. Though this is the case (Proposition 1.29), we will use (1.13) as a motivation to define the following object.

**Definition 1.28.** *For an index $\mathbf{k} = (k_1, \dots, k_r)$ and $\bullet \in \{ ш, *\}$ the* **$\bullet$-symmetric multiple zeta value** *is defined by*

$$\zeta_{\mathcal{S}}^{\bullet}(\mathbf{k}) = \zeta_{\mathcal{S}}^{\bullet}(k_1, \dots, k_r) = \sum_{j=0}^{r} (-1)^{k_1+\dots+k_j} \zeta^{\bullet}(k_j, k_{j-1}, \dots, k_1; T) \zeta^{\bullet}(k_{j+1}, \dots, k_r; T) \,,$$

*where for $j = 0$ (resp. $j = r$) we set $\zeta^{\bullet}(k_j, k_{j-1}, \dots, k_1; T) = 1$ (resp. $\zeta^{\bullet}(k_{j+1}, \dots, k_r; T) = 1$).*

In depth one we have for $k \geq 1$

$$\zeta_{\mathcal{S}}^{\bullet}(k) = \zeta^{\bullet}(k; T) + (-1)^k \zeta^{\bullet}(k; T) = \begin{cases} 2\zeta(k) & , k \text{ is even} \\ 0 & , k \text{ is odd} \end{cases} \,.$$

For the depth two case we have

$$\zeta_{\mathcal{S}}^{*}(1,1) = -\zeta(2), \qquad \zeta_{\mathcal{S}}^{ш}(1,1) = 0,$$

and for $k_1, k_2 \geq 1$ with $k_1 + k_2 \geq 3$

$$\begin{aligned} \zeta_{\mathcal{S}}^{\bullet}(k_1, k_2) &= \zeta^{\bullet}(k_1, k_2; T) + (-1)^{k_1} \zeta^{\bullet}(k_1; T) \zeta^{\bullet}(k_2; T) + (-1)^{k_1+k_2} \zeta^{\bullet}(k_2, k_1; T) \,. \\ &= (1 + (-1)^{k_1}) \zeta(k_1, k_2) + (-1)^{k_1} (1 + (-1)^{k_2}) \zeta(k_2, k_1) + (-1)^{k_1} \zeta(k_1 + k_2) \,. \end{aligned} \tag{1.14}$$

Here we used that $\zeta_{\mathcal{S}}^{*}(k_1, k_2) = \zeta_{\mathcal{S}}^{ш}(k_1, k_2)$ if $k_1 + k_2 \geq 3$. Since $(1 + (-1)^{k_1})$ vanishes for $k_1 = 1$ (same for $k_2$) there are no non-admissible indices appearing and we therefore did not need any regularization in the last equation. In particular, we see that $\zeta_{\mathcal{S}}^{\bullet}(k_1, k_2)$ also does not depend on $T$. This is the case in general and the following proposition will be proved in Section 3.

**Proposition 1.29.** *(i) We have $\zeta_{\mathcal{S}}^{\bullet}(\mathbf{k}) \in \mathcal{Z}$ for $\bullet \in \{ш, *\}$.*
*(i.e. the $\bullet$-symmetric multiple zeta values are independent of $T$).*

*(ii) For all indices $\mathbf{k}$ we have*

$$\lim_{m \to \infty} S_m(\mathbf{k}) = \zeta_{\mathcal{S}}^{*}(\mathbf{k}) \,.$$

*(iii) For all indices $\mathbf{k}$ we have*

$$\zeta_{\mathcal{S}}^{*}(\mathbf{k}) \equiv \zeta_{\mathcal{S}}^{ш}(\mathbf{k}) \mod \pi^2 \mathcal{Z} \,.$$

**Theorem 1.30.** *We have $\mathcal{Z} = \langle \zeta_{\mathcal{S}}^{*}(\mathbf{k}) \mid \mathbf{k} \text{ index} \rangle_{\mathbb{Q}} = \langle \zeta_{\mathcal{S}}^{ш}(\mathbf{k}) \mid \mathbf{k} \text{ index} \rangle_{\mathbb{Q}}$.*

This theorem is not obvious, since for example $\zeta^*_{\mathcal{S}}(k)=0$ for odd $k$, i.e. it is not obvious why $\zeta(k)$ for odd $k$ can be written as a linear combination of $\zeta^*_{\mathcal{S}}(\mathbf{k})$. But one can show (by using relations among multiple zeta values) that we have

$$\begin{aligned}
\zeta(3) &= \frac{1}{3}\zeta^*_{\mathcal{S}}(2,1)\,,\\
\zeta(5) &= \frac{2}{5}\zeta^*_{\mathcal{S}}(4,1)+\frac{2}{15}\zeta^*_{\mathcal{S}}(2,2,1)\,,\\
\zeta(7) &= \frac{10}{7}\zeta^*_{\mathcal{S}}(6,1)-\frac{3}{7}\zeta^*_{\mathcal{S}}(2,2,1,2)+\frac{5}{21}\zeta^*_{\mathcal{S}}(2,2,2,1)\,,\\
\zeta(9) &= \frac{2}{5}\zeta^*_{\mathcal{S}}(8,1)+\frac{2}{15}\zeta^*_{\mathcal{S}}(2,2,2,1,2)+\frac{4}{15}\zeta^*_{\mathcal{S}}(2,2,2,3)+\frac{4}{15}\zeta^*_{\mathcal{S}}(2,2,2,2,1)\,.
\end{aligned}$$

Moreover, Theorem 1.30 shows that relations among multiple zeta values imply relations among $*$-symmetric multiple zeta values. For example, using the extended double shuffle relations one can show that

$$2\zeta^*_{\mathcal{S}}(4,1)+\zeta^*_{\mathcal{S}}(3,2)=\zeta(5)-2\zeta(2,3)+4\zeta(4,1)=0\,.$$

Comparing this with (1.11) we see that the $\zeta^*_{\mathcal{S}}$ seem to satisfy the same relation as $\zeta_{\mathcal{A}}$. But in general we know that this is not true, since for example $\zeta^*_{\mathcal{S}}(2)=2\zeta(2)=\frac{\pi^2}{3}$ but $\zeta_{\mathcal{A}}(2)=0$. But, if we consider the $\bullet$-symmetric multiple zeta values modulo $\pi^2$, then we have that they are actually independent of $\bullet$ and they conjecturally satisfy the same relations as the finite multiple zeta values. This leads to the following definition.

**Definition 1.31.** *For* $\mathbf{k}=(k_1,\dots,k_r)$ *the* **symmetric multiple zeta value** $\zeta_{\mathcal{S}}(\mathbf{k})\in\mathcal{Z}/\pi^2\mathcal{Z}$ *is defined by*

$$\zeta_{\mathcal{S}}(\mathbf{k})\equiv\zeta^*_{\mathcal{S}}(\mathbf{k})\equiv\zeta^{ш}_{\mathcal{S}}(\mathbf{k})\mod \pi^2\mathcal{Z}.$$

Now we also have for all $k\geq 1$, that $\zeta_{\mathcal{S}}(k)=0$ (similar as for $\zeta_{\mathcal{A}}$) and the depth two case is given by the following.

**Proposition 1.32.** *For* $k_1,k_2\geq 1$ *we have*

$$\zeta_{\mathcal{S}}(k_1,k_2)\equiv\begin{cases}0, & k_1+k_2 \text{ even}\\ (-1)^{k-1}\binom{k_1+k_2}{k_1}\zeta(k_1+k_2), & k_1+k_2 \text{ odd}\end{cases}\mod \pi^2\mathcal{Z}\,.$$

*Proof.* For even $k_1+k_2\geq 2$ we get by (1.14) that

$$\zeta^*_{\mathcal{S}}(k_1,k_2)=(1+(-1)^{k_1})\zeta^*(k_1;T)\zeta^*(k_2;T)-\zeta(k_1+k_2)\,,$$

which by Proposition 1.1 is always a rational multiple of $\pi^{k_1+k_2}$. For the odd weight case we use the parity theorem, which states that for $k_1\geq 2, k_2\geq 1$ with $k_1+k_2$ odd we have (see [**?**, ])

$$\begin{aligned}
\zeta(k_1,k_2)=&(-1)^{k_2}\sum_{\substack{j=2\\ j\text{even}}}^{k-1}\left(\binom{k-j-1}{k_1-1}+\binom{k-j-1}{k_2-1}+(-1)^{k_2}\delta_{j,k_1}\right)\zeta(j)\zeta(k-j)\\
&+\frac{1}{2}\left((-1)^{k_1}\binom{k_1+k_2}{k_2}-1\right)\zeta(k)\,.
\end{aligned}$$

Plugging this into (1.14) then yields the result, since the terms in the first sum vanish modulo $\pi^2\mathcal{Z}$. $\square$

Notice that Proposition 1.32 has again strong similarities with the analogue result for finite multiple zeta values (Proposition 1.24), where we also had $\zeta_{\mathcal{A}}(k_1,k_2)=0$ for even $k_1+k_2$. For odd $k_1+k_2$ we had the same formula except that $Z(k_1+k_2)$ takes the place of $\zeta(k_1+k_2)$, which underlines the comment that $Z(k)$ (defined in 1.2) can be seen as the finite analogue of $\zeta(k)$. In general we see that, at least in the example we presented so far, the symmetric multiple zeta values satisfy the same relations as the finite multiple zeta values. This seems to be the case in general, which is part of the of the following surprising conjecture.

**Conjecture 1.33** (Kaneko-Zagier conjecture). *The following is an isomorphism of $\mathbb{Q}$-algebras*

$$\begin{aligned}\varphi_{KZ}:\mathcal{Z}^{\mathcal{A}} &\longrightarrow \mathcal{Z}/\pi^2\mathcal{Z}\\ \zeta_{\mathcal{A}}(\mathbf{k}) &\longmapsto \zeta_{\mathcal{S}}(\mathbf{k})\,.\end{aligned}$$

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In analogy, and in accordance with Conjecture 1.33, we also have the following dimension conjecture.

**Conjecture 1.34** (Zagier). *For $k\geq 3$ we have $\dim_{\mathbb{Q}}\mathcal{Z}_k^{\mathcal{A}}=d_{k-3}$.*

Conjecture (1.33) in particular predicts that the finite and symmetric multiple zeta values satisfy the same $\mathbb{Q}$-linear relations. There are various families of linear relations which have been proven for finite as well as for symmetric multiple zeta values. One of these are given by the Hoffman duality. To state it we first introduce the star-versions of finite and symmetric multiple zeta values.
For $\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}^r$ and $m\geq 1$ define

$$H^{\star}{}_m(\mathbf{k})=H^{\star}{}_m(k_1,\dots,k_r)=\sum_{m\geq m_1\geq\dots\geq m_r>0}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\in\mathbb{Q}\,, \tag{1.15}$$

and the **finite multiple zeta star-values**

$$\zeta^{\star}_{\mathcal{A}}(\mathbf{k})=\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_r)=\left(H^{\star}_{p-1}(\mathbf{k})\mod p\right)_p=\left(\sum_{p>m_1\geq\dots\geq m_r>0}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}\mod p\right)_p\in\mathcal{A}\,.$$

Here we also set $\zeta^{\star}_{\mathcal{A}}(\emptyset)=1$. Notice that the star-values of both objects are just a linear combination of the usual values, since for example

$$\zeta^{\star}_{\mathcal{A}}(k_1,k_2,k_3)=\zeta_{\mathcal{A}}(k_1,k_2,k_3)+\zeta_{\mathcal{A}}(k_1+k_2,k_3)+\zeta_{\mathcal{A}}(k_1,k_2+k_3)+\zeta_{\mathcal{A}}(k_1+k_2+k_3)\,.$$

In general we have

$$\zeta^{\star}_{\mathcal{A}}(k_1,\dots,k_r)=\sum_{\circ_j=\text{`}+\text{` or }\circ_j=\text{`},\text{`}}\zeta_{\mathcal{A}}(k_1\circ_1\cdots\circ_{r-1}k_r)\,.$$

Using this description we define the **star-version of the symmetric multiple zeta values** by

$$\zeta^{\star}_{\mathcal{S}}(k_1,\dots,k_r)=\sum_{\circ_j=\text{`}+\text{` or }\circ_j=\text{`},\text{`}}\zeta_{\mathcal{S}}(k_1\circ_1\cdots\circ_{r-1}k_r)\in\mathcal{Z}/\pi^2\mathcal{Z}\,.$$

By (1.33) the $\zeta^{\star}_{\mathcal{A}}$ and $\zeta^{\star}_{\mathcal{S}}$ should therefore also satisfy the same relations.

Every index $\mathbf{k}=(k_1,\dots,k_r)$ can be written as

$$(k_1,\dots,k_r)=(\overbrace{1+\dots+1}^{k_1},\dots,\overbrace{1+\dots+1}^{k_r}).$$

Define the **Hoffman dual** $\mathbf{k}^\vee$ by interchanging $,$ and $+$ in this representation.
For example the Hoffman dual of $\mathbf{k}=(3,2)$ is given by

$$\mathbf{k}^\vee=(3,2)^\vee=(1+1+1,1+1)^\vee=(1,1,1+1,1)=(1,1,2,1).$$

With this we have the following family of linear relation which is true for finite as well as symmetric multiple zeta values.

**Theorem 1.35.** *[ {♨ **Todo:** references♨}] For all indices $\mathbf{k}$ and $\mathcal{F}\in\{\mathcal{A},\mathcal{S}\}$ we have*

$$\zeta^\star_{\mathcal{F}}(\mathbf{k})=-\zeta^\star_{\mathcal{F}}(\mathbf{k}^\vee).$$

We will see that Theorem 1.35 can also be proven by using the results in [BTT].

## 1.3 $q$-analogues of multiple zeta values

A $q$-analogue of a theorem, identity or expression is a generalization involving a new parameter $q$ that returns the original theorem, identity or expression in the limit as $q\to 1$[2]. The easiest example is the $q$-analogue of a natural number $m$ given by the $q$-integer

$$[m]_q=\frac{1-q^m}{1-q}=1+q+\dots+q^{m-1},\qquad \lim_{q\to1}[m]_q=m. \tag{1.16}$$

There are various different models of $q$-analogues of multiple zeta values in the literature. We will consider a few of them in this course and start with the most common model which was first independently studied by Bradley [Bra] and Zhao [Zh2]. For an admissible index $\mathbf{k}=(k_1,\dots,k_r)$ these are defined by

$$\zeta_q^{\mathrm{BZ}}(\mathbf{k})=\zeta_q(\mathbf{k})=\zeta_q(k_1,\dots,k_r)=\sum_{m_1>\dots>m_r>0}\frac{q^{(k_1-1)m_1}\cdots q^{(k_r-1)m_r}}{[m_1]_q^{k_1}\cdots[m_r]_q^{k_r}}.$$

Here the $q$ can be viewed as a formal parameter, i.e. $\zeta(\mathbf{k};q)\in\mathbb{Q}[[q]]$, or we can view $q$ as a complex number with $|q|<1$. We will not really distinguish between these two point of views since all our objects can be viewed as holomorphic functions in the open unit disc or as formal power series with rational coefficients. By (1.16), together with a justification that one can interchange summation and the limit, we have

$$\lim_{q\to1}\zeta(\mathbf{k};q)=\zeta(\mathbf{k}).$$

In Section **??** we will see that these $q$-series satisfy a lot of relations which are satisfied by multiple zeta values. In particular, this model also satisfies the relation $\zeta(2,1;q)=\zeta(3;q)$.

But they do not satisfy exactly the same stuffle/harmonic product formula, since we have

$$\frac{q^{(k_1-1)m}}{[m]^{k_1}}\frac{q^{(k_2-1)m}}{[m]^{k_2}}=\frac{q^{(k_1+k_2-1)m}}{[m]_q^{k_1+k_2}}+(1-q)\frac{q^{(k_1+k_2-2)m}}{[m]_q^{k_1+k_2-1}},$$

[2]Here and in the following we mean by $q\to1$ the limit of $q$ to 1 on the real axis with $|q|<1$.

which implies, for example in small depth for $k_1, k_2 \geq 2$,

$$\zeta_q(k_1)\zeta_q(k_2) = \zeta_q(k_1,k_2) + \zeta_q(k_2,k_1) + \zeta_q(k_1+k_2) + (1-q)\zeta_q(k_1+k_2-1)\,. \tag{1.17}$$

In particular this shows, that the $\mathbb{Q}$-vector space spanned by all $\zeta_q$ is not a $\mathbb{Q}$-algebra (but a $\mathbb{Q}(1-q)$-algebra). For some applications, such as the connection to modular forms (see [B6] for details), it is convenient to consider a **modified version** of $q$-analogues. In the case of the Bradley-Zhao model this is given by

$$\overline{\zeta}_q^{\mathrm{BZ}}(\mathbf{k}) = \overline{\zeta}_q(\mathbf{k}) = \overline{\zeta}_q(k_1,\dots,k_r) = \sum_{m_1>\dots>m_r>0} \frac{q^{(k_1-1)m_1}\cdots q^{(k_r-1)m_r}}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_r})^{k_r}}\,.$$

In this case we have for an admissible index

$$\lim_{q\to 1}(1-q)^{\mathrm{wt}(\mathbf{k})}\overline{\zeta}(\mathbf{k};q) = \zeta(\mathbf{k})\,.$$

and

$$\overline{\zeta}(k_1)\overline{\zeta}_q(k_2) = \overline{\zeta}_q(k_1,k_2) + \overline{\zeta}_q(k_2,k_1) + \overline{\zeta}_q(k_1+k_2) + \overline{\zeta}_q(k_1+k_2-1)\,. \tag{1.18}$$

There are various different models of (modified) $q$-analogues and later we will consider general sums of the form

$$\sum_{m_1>\dots>m_r>0} \frac{Q_1(q^{m_1})\cdots Q_r(q^{m_r})}{(1-q^{m_1})^{k_1}\cdots(1-q^{m_r})^{k_r}}\,.$$

for polynomials $Q_j(X) \in \mathbb{Q}[X]$.

## 1.4 The BTT-philosophy

{☕ **Todo:** Some general explanation about [BTT] ☕}

In analogy to the multiple harmonic sum we define the **multiple harmonic $q$-sum**, for $\mathbf{k} = (k_1,\dots,k_r) \in \mathbb{Z}^r$ and $m \geq 1$ by

$$H_m(\mathbf{k};q) = H_m(k_1,\dots,k_r;q) = \sum_{m\geq m_1>\dots>m_r>0} \frac{q^{(k_1-1)m_1}\cdots q^{(k_r-1)m_r}}{[m_1]_q^{k_1}\cdots[m_r]_q^{k_r}} \in \mathbb{Q}[[q]]\,. \tag{1.19}$$

Notice that for an admissible index $\mathbf{k}$ we have

$$\lim_{q\to 1}\lim_{m\to\infty} H_m(\mathbf{k};q) = \lim_{q\to 1}\zeta_q(\mathbf{k}) = \zeta(\mathbf{k})\,.$$

But some kind of magic happens, when on considers these limits in some sense at the same time. By this we mean that we make $q$ dependent on $m$ and then send $m\to\infty$. For this we consider the values $H_{n-1}(\mathbf{k};\zeta_n) \in \mathbb{Q}(\zeta_n)$, where now $q=\zeta_n$ is a primitive $n$-th root of unity. For $n\to\infty$ we then get $\zeta_n \to 1$, i.e. in some sense we consider $q\to 1$ and $n\to\infty$ at the same time. Doing this for the explicit $n$-th root of unity $\zeta_n = e^{\frac{2\pi i}{n}}$ leads to the following result.

**Theorem 1.36.** *For any index set $\mathbf{k} = (k_1,\dots,k_r)$ the limit $\lim_{n\to\infty} H_{n-1}(\mathbf{k}; e^{\frac{2\pi i}{n}})$ exists and we set*

$$\xi(\mathbf{k}) := \lim_{n\to\infty} H_{n-1}(\mathbf{k}; e^{\frac{2\pi i}{n}}) \in \mathbb{C}\,.$$

*It is given by*

$$\xi(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1+\dots+k_a} \zeta^*\big(k_a, k_{a-1}, \dots, k_1; \frac{\pi i}{2}\big) \zeta^*\big(k_{a+1}, k_{a+2}, \dots, k_r; -\frac{\pi i}{2}\big)\,.$$

*In particular, we have*

$$\operatorname{Re}(\xi(\mathbf{k})) \equiv \zeta_{\mathcal{S}}(\mathbf{k}) \mod \pi^2 \mathcal{Z}\,.$$

**Theorem 1.37.** *For any primitive $p$-th root of unity $\zeta_p$, we have*

$$(H_{p-1}(\mathbf{k}; \zeta_p) \mod \mathfrak{p})_p = \zeta_{\mathcal{A}}(\mathbf{k})\,,$$

*where $\mathfrak{p} = (1-\zeta_p)$ is the prime ideal of $\mathbb{Z}[\zeta_p]$ generated by $1-\zeta_p$.*

*Proof.* For $p$ prime we have $H_{p-1}(\mathbf{k}) \in \mathbb{Z}[\zeta_p]$. This follows from the fact that the $q$-integer $[m]_q$ at $q = \zeta_p$ is a cyclotomic unit, when $m$ is coprime with $p$, since in this case there exists a $t$ with $m \cdot t \equiv 1 \mod p$ and therefore

$$\frac{1}{[m]_{\zeta_p}} = \frac{1-\zeta_p}{1-\zeta_p^m} = \frac{1-\zeta_p^{tm}}{1-\zeta_p^m} = 1 + \zeta_p^m + \dots + \zeta_p^{(t-1)m} \in \mathbb{Z}[\zeta_p]\,.$$

Moreover, we have $\mathbb{Z}[\zeta_p]/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$ (Exercise 6) and for $p > m > 0$ we have $[m]_{\zeta_p} \equiv m \mod \mathfrak{p}$. Combining all these shows the desired result. □

{☕ **Todo:** include picture to illustrate BTT philosophy ☕}

🐱——— Until here in lecture 4 (13th May, 2022) ———🐱

**Goal:** Understand the values $H_{n-1}(\mathbf{k}; \zeta_n)$ for primitive $n$-th root of unities $\zeta_n$ (e.g. $\zeta_n = e^{\frac{2\pi i}{n}}$) and their relations to get simultaneous results on finite and symmetric multiple zeta values.

For this we start by considering the depth one case again. Carlitz (1956) introduced the **degenerated Bernoulli** numbers $b_k(n) \in \mathbb{Q}[\frac{1}{n}]$

$$\sum_{k=0}^{\infty} b_k(n) \frac{x^k}{k!} = \frac{x}{(1+\frac{x}{n})^n - 1}\,.$$

These numbers can be seen as a degeneration of the Bernoulli numbers

$$\lim_{n\to\infty} b_k(n) = B_k\,, \qquad \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}\,.$$

For even $k \geq 2$ we saw that $\zeta(k) = -\frac{B_k}{k!}\frac{(2\pi i)^k}{2}$ and as an analogue we have

**Proposition 1.38.** *For all $n, k \geq 1$ we have*

$$H_{n-1}(k; e^{\frac{2\pi i}{n}}) = -\frac{b_k(n)}{k!}\Big(n(1-e^{\frac{2\pi i}{n}})\Big)^k\,.$$

*Proof.* This is Exercise 7. □

For an index set $\mathbf{k} = (k_1, \dots, k_r)$ we define its reverse by $\overline{\mathbf{k}} = (k_r, \dots, k_1)$.

**Theorem 1.39.** *For all $n \geq 1$ and all indices $\mathbf{k}$ we have for any primitive $n$-th root of unity $\zeta_n$*

$$H^\star_{n-1}(\mathbf{k}; \zeta_n) = (-1)^{\mathrm{wt}(\mathbf{k})+1} H^\star_{n-1}(\overline{\mathbf{k}^\vee}; \zeta_n)\,.$$

Theorem 1.35 ($\zeta^\star_{\mathcal{A}}(\mathbf{k}) = -\zeta^\star_{\mathcal{A}}(\mathbf{k}^\vee)$) follows from this together with the reversal relation $\zeta^\star_{\mathcal{A}}(\mathbf{k}) = (-1)^{\mathrm{wt}(\mathbf{k})}\zeta^\star_{\mathcal{A}}(\overline{\mathbf{k}})$.

# §2 Algebraic setup & ($q$-)Multiple polylogarithm

## 2.1 Quasi-shuffle algebras

In this section, we want to describe the algebraic setup for (finite/symmetric) multiple zeta values and their $q$-analogues.

In the following, we assume that $k$ is a field containing $\mathbb{Q}$, and $A$ is a countable set to which we refer as the **set of letters**. Let $kA$ be the $k$-vector space generated by $A$ and let $\diamond$ be a $k$-bilinear, associative and commutative product on $kA$. We obtain a (non-unital) $k$-algebra $(kA, \diamond)$, to which we refer as the **algebra of letters**. Notice that in the work [HI] the authors just assume that $\diamond$ is associative and commutative and do not consider $kA$ as an algebra, but it will be useful for our purposes (e.g., Lemma 2.4). For such a product $\diamond$ on letters, we want to assign a product $*_\diamond$ on the space of words $k\langle A\rangle$, which generalizes the stuffle and shuffle product we have seen before. Here a monomial $w = a_1 \cdots a_l$ in $k\langle A\rangle = k\langle a_1, a_2, \dots\rangle$ will again be called a **word** and the unit ($l = 0$), denoted by 1, is again called the **empty word**. By $\ell(w) = l$ we denote the **length of the word** $w$.

**Definition 2.1.** *Let $\diamond$ be a product on $kA$ as above. Then we define the* **quasi-shuffle product** $*_\diamond$ *on $k\langle A\rangle$ as the $k$-bilinear product, which satisfies $1 *_\diamond w = w *_\diamond 1 = w$ for any word $w \in k\langle A\rangle$ and*

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a \diamond b)(w *_\diamond v)$$

*for any letters $a, b \in A$ and words $w, v \in k\langle A\rangle$.*

**Theorem 2.2.** *The space $k\langle A\rangle$ equipped with the product $*_\diamond$ becomes a commutative $k$-algebra.*

*Proof.* This is Theorem 2.1 in [HI]. It suffices to show that $*_\diamond$ is commutative and associative, which can be done straightforward by induction on the lengths of words. (Exercise 8) □

We will call $(k\langle A\rangle, *_\diamond)$ a **quasi-shuffle algebra** or **algebra of words**.

*Examples* 2.3. We will now give the main example for quasi-shuffle algebras used in this course.

(i) For any alphabet $A$ one can consider the trivial product $a \diamond b = 0$ for all $a, b \in A$. The induced quasi-shuffle product is the usual **shuffle product** and we write $\sqcup\!\sqcup = *_\diamond$. In particular, we consider the shuffle product for the alphabet $A_{xy} = \{x, y\}$ and we set

$$\mathfrak{H} = \mathbb{Q}\langle A_{xy}\rangle = \mathbb{Q}\langle x, y\rangle\,.$$

The corresponding quasi-shuffle algebra will be denoted by $\mathfrak{H}_{\sqcup\!\sqcup} = (\mathfrak{H}, \sqcup\!\sqcup)$. Explicitly the $\sqcup\!\sqcup$ is on $\mathfrak{H}$ defined as the $\mathbb{Q}$-bilinear product, which satisfies $1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1 = w$ for any word $w \in \mathfrak{H}$ and

$$a_1 w_1 \sqcup\!\sqcup a_2 w_2 = a_1(w_1 \sqcup\!\sqcup a_2 w_2) + a_2(a_1 w_1 \sqcup\!\sqcup w_2) \tag{2.1}$$

for any letters $a_1, a_2 \in \{x, y\}$ and words $w_1, w_2 \in \mathfrak{H}$.

The space $\mathfrak{H}$ has two natural subspaces $\mathfrak{H}^1$ and $\mathfrak{H}^0$ defined by

$$\mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}y\,,$$
$$\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y \subset \mathfrak{H}^1\,.$$

Here $\mathfrak{H}^1$ is the set of linear combinations of words ending in $y$ (and the empty word) and $\mathfrak{H}^0$ is the set of linear combinations of words starting in $x$ and ending in $y$. By (2.1) it is easy to see that both $\mathfrak{H}^1$ and $\mathfrak{H}^0$ are closed under $⧢$ and we therefore get the following inclusion of $\mathbb{Q}$-algebras

$$\mathfrak{H}^0_{⧢} \subset \mathfrak{H}^1_{⧢} \subset \mathfrak{H}_{⧢}\,.$$

(ii) The next important example is the alphabet $A_z = \{z_1, z_2, \dots\}$ and the product $z_{k_1} \diamond z_{k_2} = z_{k_1+k_2}$. The corresponding quasi-shuffle product will be denoted by $* = *_\diamond$ and is called the **stuffle product** (or **harmonic product**).

Notice that via the identification $z_k \leftrightarrow \overbrace{x\dots x}^{k-1}y$ the space $\mathfrak{H}^1$ discussed in (i) is isomorphic to $\mathbb{Q}\langle A_z\rangle = \mathbb{Q}\langle z_1, z_2, \dots\rangle$. By abuse of notation we write $\mathfrak{H}^1 = \mathbb{Q}\langle A_z\rangle$ and denote the corresponding quasi-shuffle algebra by $\mathfrak{H}^1_* = (\mathfrak{H}^1, *)$. The stuffle product $*$ on $\mathfrak{H}^1$ is explicitly given as the $\mathbb{Q}$-bilinear product, which satisfies $1 * w = w * 1 = w$ for any word $w \in \mathfrak{H}^1$ and

$$z_i w_1 * z_j w_2 = z_i(w_1 * z_j w_2) + z_j(z_i w_1 * w_2) + z_{i+j}(w_1 * w_2)$$

for any $i, j \geq 1$ and words $w_1, w_2 \in \mathfrak{H}^1$. Notice that $\mathfrak{H}^0$ is also closed under $*$ and we obtain the following inclusion of $\mathbb{Q}$-algebras

$$\mathfrak{H}^0_* \subset \mathfrak{H}^1_*.$$

For an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{\geq 1}$ we define

$$z_{\mathbf{k}} = z_{k_1} z_{k_2} \cdots z_{k_r} \in \mathfrak{H}^1$$

and set $z_\emptyset = 1$. Now define the space $\mathfrak{H}^0 = \mathbb{Q} + x\mathfrak{H}y$, which is the subspace of $\mathfrak{H}^1$ generated by all words which start in $x$ and end in $y$. In other words, $\mathfrak{H}^0$ is spanned by all $z_{\mathbf{k}}$ with admissible indices $\mathbf{k}$. Summarizing everything we have

$$\mathfrak{H}^0 = \langle z_{\mathbf{k}} \mid \mathbf{k} \text{ admissible index}\rangle_{\mathbb{Q}} \quad \subset \quad \mathfrak{H}^1 = \langle z_{\mathbf{k}} \mid \mathbf{k} \text{ index}\rangle_{\mathbb{Q}} \quad \subset \quad \mathfrak{H} = \mathbb{Q}\langle x, y\rangle\,.$$

(iii) The next quasi-shuffle product is inspired by the product (1.17) among $\zeta_q$ and the product (1.18) among their modified versions $\overline{\zeta}_q$. We use again the alphabet $A_z$, but now consider the field $k = \mathbb{Q}(\hbar)$ for a formal variable $\hbar$ (we will use $\hbar = 1 - q$ for $\zeta_q$ and $\hbar = 1$ for the modified version). Then we define $z_{k_1} \diamond_q z_{k_2} = z_{k_1+k_2} + \hbar z_{k_1+k_2-1}$ and denote the corresponding quasi-shuffle product by $*_q = *_{\diamond_q}$.

**Lemma 2.4.** *Let $R$ be a $k$-algebra and $f_m\colon (kA, \diamond) \to R$ be $k$-algebra homomorphisms for $m \geq 1$. Then for all $M \geq 1$ the $k$-linear map $F_M : k\langle A\rangle \to R$ defined on a word $w = a_1 \cdots a_r \in k\langle A\rangle$ by*

$$F_M(w) = \sum_{M > m_1 > \dots > m_r > 0} f_{m_1}(a_1) \cdots f_{m_r}(a_r)$$

*and $F_M(1) = 1$ is a $k$-algebra homomorphism from $(k\langle A\rangle, *_\diamond)$ to $R$.*

*Examples* 2.5. For the quasi-shuffle products defined in 2.3 (ii) and (iii) we will consider the following examples:

(i) Let $k=\mathbb{Q}$,$\mathbb{R}=\mathbb{Q}$ and define for $m\geq 1$ the map

$$\begin{aligned} f_m : \mathbb{Q}A_z &\longrightarrow \mathbb{Q} \\ z_k &\longmapsto \frac{1}{m^k}\,. \end{aligned}$$

{♨ **Todo:** include the other examples♨}

*Proof of Lemma 2.4* . It suffices to show that for any $M\geq 1$ and words $w,v\in k\langle A\rangle$ we have

$$F_M(w)F_M(v)=F_M(w *_\diamond v)\,.$$

We will prove this by induction on $M$. The case $M=1$ is trivial, since $F_1(w)=0$ for all non-empty $w$ and $1=F_1(1)F_1(1)=F_1(1 *_\diamond 1)$. Notice that we have $F_M(aw)=\sum_{M>m>0} f_m(a)F_m(w)$ for a letter $a$ and a word $w$. For $w=aw'$, $v=bv'$ with letters $a,b\in A$ and words $w',v'\in k\langle A\rangle$ we therefore get

$$\begin{aligned} F_M(w)F_M(v) &= \sum_{M>m>0} f_m(a)F_m(w') \sum_{M>n>0} f_n(b)F_n(v') \\ &= \left(\sum_{M>m>n>0} + \sum_{M>n>m>0} + \sum_{M>m=n>0}\right) f_m(a)F_m(w')f_n(b)F_n(v') \\ &= \sum_{M>m>0} f_m(a)F_m(w')F_m(bv') + \sum_{M>n>0} f_n(b)F_n(aw')F_n(v') + \sum_{M>m>0} f_m(a)f_m(b)F_m(w')F_m(v') \\ &= \sum_{M>m>0} f_m(a)F_m(w' *_\diamond bv') + \sum_{M>n>0} f_n(b)F_n(aw' *_\diamond v') + \sum_{M>m>0} f_m(a\diamond b)F_m(w' *_\diamond v') \\ &= F_M(a(w' *_\diamond bv')) + F_M(b(aw' *_\diamond v')) + F_M((a\diamond b)(w' *_\diamond v')) = F_M(w *_\diamond v)\,. \end{aligned}$$

Here we used that $f_m$ is an algebra homomorphism together with the induction hypothesis in the fourth equation. □

{♨ **Todo:** Maybe give a generalization of Lemma 2.4 for arbitrary total orders for the sums $S_n$.♨}

Until here in lecture 5 (27th May, 2022)

Define the $\mathbb{Q}$-linear map from $\mathfrak{H}^0$ to the space of multiple zeta values $\mathcal{Z}$, defined on the generators by

$$\begin{aligned} \zeta\colon \mathfrak{H}^0 &\longrightarrow \mathcal{Z} \\ z_{\mathbf{k}} &\longmapsto \zeta(\mathbf{k})\,. \end{aligned}$$

We write $\zeta\colon w\mapsto\zeta(w)$ for any $w\in\mathfrak{H}^0$. As a consequence of Lemma 2.4 we get the following.

**Proposition 2.6.** *For any $w,u\in\mathfrak{H}^0$ we have*

$$\zeta(w)\zeta(v)=\zeta(w*v)\,.$$

*In particular the space $\mathcal{Z}$ is a $\mathbb{Q}$-subalgebra of $\mathbb{R}$ and $\zeta$ is an algebra homomorphism from $\mathfrak{H}^0_*$ to $\mathcal{Z}$.*

Similar statements are true for $\zeta_{\mathcal{A}}$,$\zeta_{\mathcal{S}}$,$H_M$, $\zeta_q$. {♨ **Todo:** make this precise♨}

## 2.2 Multiple polylogarithms & finite double shuffle relations

In the following we want to introduce the iterated integrals expression for multiple zeta values. This will be used to show that multiple zeta values give rise to algebra homomorphisms from $\mathfrak{H}^0_{ш}$ to $\mathcal{Z}$. We start by calculating one simple example by hand, before giving a general formula afterwards. Consider the following iterated integral

$$\begin{aligned}\int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} &= \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \sum_{n=0}^{\infty} t_2^n dt_2 = \int_0^1 \frac{dt_1}{t_1} \left[ \sum_{n=0}^{\infty} \frac{t_2^{n+1}}{n+1} \right]_0^{t_1} \\ &= \int_0^1 \sum_{n=0}^{\infty} \frac{t_1^n}{n+1} dt_1 = \left[ \sum_{n=0}^{\infty} \frac{t_1^{n+1}}{(n+1)^2} \right]_0^1 = \sum_{m>0} \frac{1}{m^2} = \zeta(2)\,. \end{aligned} \tag{2.2}$$

With the same idea one can also show that we have (Exercise 6 i))

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}\,. \tag{2.3}$$

In general we will see that an index $\mathbf{k} = (k_1, \dots, k_r)$ correspondonds to an iterated integral of length $\mathrm{wt}(\mathbf{k})$, where each $k_j$ gives a block of $k_j - 1$ integrals over $\frac{dt}{t}$ and one integral over $\frac{dt}{1-t}$. To prove these iterated integrals in general we will introduce multiple polylogarithms, which can be seen as a simultaneous generalization of the polylogarithm ($r = 1$) and multiple zeta values ($z = 1$).

**Definition 2.7.** *For $|z| < 1$ and $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{\geq 1}$ we define the* **multiple polylogarithm** *and their star versions by*

$$\begin{aligned}\mathrm{Li}_{\mathbf{k}}(z) = \mathrm{Li}_{k_1,\dots,k_r}(z) &= \sum_{m_1 > \cdots > m_r > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}} \\ \mathrm{Li}^\star_{\mathbf{k}}(z) = \mathrm{Li}^\star_{k_1,\dots,k_r}(z) &= \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{z^{m_1}}{m_1^{k_1} \cdots m_r^{k_r}}\end{aligned}$$

*and set $\mathrm{Li}_\emptyset(z) = \mathrm{Li}^\star_\emptyset(z) = 1$.*

For an arbitrary index $\mathbf{k}$ the $\mathrm{Li}_{\mathbf{k}}(z)$ are holomorphic functions in the open unit disc, but clearly when $\mathbf{k}$ is admissible $\mathrm{Li}_{\mathbf{k}}(z)$ is also defined for $z = 1$ and we have

$$\mathrm{Li}_{\mathbf{k}}(1) = \zeta(\mathbf{k})\,.$$

Multiple polylogarithms also have an iterated integral expression, and for example using the same calculation as in (2.2) we see for example that

$$\mathrm{Li}_2(z) = \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2}\,.$$

From now on we will restrict to real $|z| < 1$ and consider integrals on the real axis. Since $\mathrm{Li}_{\mathbf{k}}(z)$ is defined for any $\mathbf{k}$, we can view Li as a $\mathbb{Q}$-linear map from $\mathfrak{H}^1$ to the space of real valued continuous functions on $(0,1)$, i.e. $C((0,1);\mathbb{R})$, defined on the generators by

$$\begin{aligned}\mathrm{Li}\colon \mathfrak{H}^1 &\longrightarrow C((0,1);\mathbb{R}) \\ z_{\mathbf{k}} &\longmapsto \mathrm{Li}_{\mathbf{k}}(z)\,.\end{aligned}$$

By abuse of notation we write $\mathrm{Li}\colon w \mapsto \mathrm{Li}_w(z)$ for any $w \in \mathfrak{H}^1$, which is defined by linearly extending the definition on the generators $z_{\mathbf{k}}$. For example for $w = xyxxy + 2xxxxy = z_2z_3 + 2z_5 \in \mathfrak{H}^1$ we have $\mathrm{Li}_w(z) = \mathrm{Li}_{2,3}(z) + 2\,\mathrm{Li}_5(z)$. Now we want to describe the iterated integral expression for the multiple polylogarithm using this setup.

**Lemma 2.8.** *Let $w \in \mathfrak{H}^1$ be a linear combination of words all starting with the letter $a \in \{x,y\}$, i.e. $w = au$ for some $u \in \mathfrak{H}^1$. Then we have*

$$\frac{d}{dz}\mathrm{Li}_w(z) = \frac{d}{dz}\mathrm{Li}_{au}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_u(z)\,, & a = x \\ \frac{1}{1-z}\mathrm{Li}_u(z)\,, & a = y \end{cases}\,,$$
$$\frac{d}{dz}\mathrm{Li}^\star_w(z) = \frac{d}{dz}\mathrm{Li}^\star_{au}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}^\star_u(z)\,, & a = x \\ \left(\frac{1}{z} + \frac{1}{1-z}\right)\mathrm{Li}^\star_u(z)\,, & a = y \end{cases}\,.$$

*Proof.* Since Li is linear it suffices to proof the statement for a word $w$. Assuming $w = z_{\mathbf{k}}$ for $\mathbf{k} = (k_1,\dots,k_r)$, we have

$$\frac{d}{dz}\mathrm{Li}_w(z) = \frac{d}{dz}\mathrm{Li}_{\mathbf{k}}(z) = \frac{d}{dz}\sum_{m_1>\dots>m_r>0}\frac{z^{m_1}}{m_1^{k_1}m_2^{k_2}\cdots m_r^{k_r}} = \sum_{m_1>\dots>m_r>0}\frac{z^{m_1-1}}{m_1^{k_1-1}m_2^{k_2}\cdots m_r^{k_r}}\,.$$

Let $a = x$, which is equivalent to $k_1 > 1$. In this case we obtain

$$\frac{d}{dz}\mathrm{Li}_w(z) = \frac{d}{dz}\mathrm{Li}_{xu}(z) = \frac{1}{z}\sum_{m_1>\dots>m_r>0}\frac{z^{m_1}}{m_1^{k_1-1}m_2^{k_2}\cdots m_r^{k_r}} = \frac{1}{z}\mathrm{Li}_u(z)\,.$$

If $a = y$, then we have $k_1 = 1$ and

$$\begin{aligned}\frac{d}{dz}\mathrm{Li}_w(z) = \frac{d}{dz}\mathrm{Li}_{yu}(z) &= \sum_{m_1>\dots>m_r>0}\frac{z^{m_1-1}}{m_2^{k_2}\cdots m_r^{k_r}} = \sum_{m_2>\dots>m_r>0}\frac{1}{m_2^{k_2}\cdots m_r^{k_r}}\sum_{m_1=m_2+1}^{\infty} z^{m_1-1} \\ &= \frac{1}{1-z}\sum_{m_2>\dots>m_r>0}\frac{z^{m_2}}{m_2^{k_2}\cdots m_r^{k_r}} = \frac{1}{1-z}\mathrm{Li}_{k_2,\dots,k_r}(z) = \frac{1}{1-z}\mathrm{Li}_u(z)\,.\end{aligned}$$

{☕ **Todo:** prove the star version☕} □

Motivated by the iterated integrals (2.2), (2.3), and the above Lemma, we define

$$\omega_x(t) = \frac{dt}{t}\,, \qquad \omega_y(t) = \frac{dt}{1-t}\,.$$

With these differential forms we can write the multiple polylogarithms as the following iterated integral.

**Proposition 2.9.** *For any word $w = a_1 \dots a_k \in \mathfrak{H}^1$, with $a_1,\dots,a_k \in \{x,y\}$ and $0 \le z < 1$ we have*

$$\mathrm{Li}_w(z) = \int_0^z \omega_{a_1}(t_1)\int_0^{t_1}\omega_{a_2}(t_2)\cdots\int_0^{t_{k-1}}\omega_{a_k}(t_k)\,.$$

*Proof.* This follows from Lemma 2.8 by induction on $k$. In the case $k = 1$ we have $w = y = z_1$, i.e.

$$\mathrm{Li}_w(z) = \mathrm{Li}_1(z) = \sum_{m>0}\frac{z^m}{m} = \int_0^z \frac{dt}{1-t} = \int_0^z \omega_y(t)\,.$$

The induction step is then exactly the statement of Lemma 2.8 since $\mathrm{Li}_w(0) = 0$ for non-empty $w$. □



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For a real $z$ we will also use the following simplified notation for iterated integrals for $a_1 \dots a_k \in \mathfrak{H}^1$

$$\int\limits_{z>t_1>\dots>t_k>0} \omega_{a_1}(t_1)\cdots\omega_{a_k}(t_k) := \int_0^z \omega_{a_1}(t_1)\int_0^{t_1}\omega_{a_2}(t_2)\cdots\int_0^{t_{k-1}}\omega_{a_k}(t_k)\,.$$

Since $\mathrm{Li}_w(1)=\zeta(w)$ for any $w\in\mathfrak{H}^0$ we also get an iterated integral expression for multiple zeta values as a consequence of Proposition 2.9.

**Corollary 2.10.** *For any word $w = a_1 \dots a_k \in \mathfrak{H}^0$, with $a_1,\dots,a_k\in\{x,y\}$ we have*

$$\zeta(w) = \int\limits_{1>t_1>\dots>t_k>0} \omega_{a_1}(t_1)\cdots\omega_{a_k}(t_k)\,.$$

**Proposition 2.11.** *For any $w,u\in\mathfrak{H}^1$ we have*

$$\mathrm{Li}_w(z)\,\mathrm{Li}_u(z) = \mathrm{Li}_{w\sqcup\!\sqcup u}(z)\,,$$

*i.e. the map* Li *is an algebra homomorphism from $\mathfrak{H}^1_{\sqcup\!\sqcup}$ to $C((0,1);\mathbb{R})$.*

*Proof.* It is sufficient to prove the statement for words $w,u\in\mathfrak{H}^1$. We will do this by induction on the sum of the lengths of $w$ and $u$. If one of them equals the empty word 1, the statement is clear. So lets assume that $w=aw'$ and $u=bu'$ for words $w',u'\in\mathfrak{H}^1$ and letters $a,b\in\{x,y\}$. Then we have

$$\frac{d}{dz}\left(\mathrm{Li}_w(z)\,\mathrm{Li}_u(z)\right) = \frac{d}{dz}\left(\mathrm{Li}_{aw'}(z)\,\mathrm{Li}_{bu'}(z)\right) = \left(\frac{d}{dz}\mathrm{Li}_{aw'}(z)\right)\mathrm{Li}_{bu'}(z) + \mathrm{Li}_{aw'}(z)\left(\frac{d}{dz}\mathrm{Li}_{bu'}(z)\right).$$

Using now Lemma 2.8 we get $\frac{d}{dz}\mathrm{Li}_{aw'}(z) = f_a(z)\,\mathrm{Li}_{w'}(z)$ with $f_x(z)=\frac{1}{z}$ and $f_y(z)=\frac{1}{1-z}$. Using this together with the induction hypothesis we have

$$\frac{d}{dz}\left(\mathrm{Li}_w(z)\,\mathrm{Li}_u(z)\right) = f_a(z)\,\mathrm{Li}_{w'}(z)\,\mathrm{Li}_{bu'}(z) + f_b(z)\,\mathrm{Li}_{aw'}(z)\,\mathrm{Li}_{u'}(z) = f_a(z)\,\mathrm{Li}_{w'\sqcup\!\sqcup bu'}(z) + f_b(z)\,\mathrm{Li}_{aw'\sqcup\!\sqcup u'}(z)\,.$$

Applying Lemma 2.8 again gives

$$\frac{d}{dz}\left(\mathrm{Li}_w(z)\,\mathrm{Li}_u(z)\right) = \frac{d}{dz}\mathrm{Li}_{a(w'\sqcup\!\sqcup bu')}(z) + \frac{d}{dz}\mathrm{Li}_{b(aw'\sqcup\!\sqcup u')}(z) = \frac{d}{dz}\mathrm{Li}_{w\sqcup\!\sqcup u}(z)\,,$$

i.e. $\mathrm{Li}_w(z)\,\mathrm{Li}_u(z) = \mathrm{Li}_{w\sqcup\!\sqcup u}(z) + c$ for some constant $c$. But since both sides vanish for $z=0$, we conclude $c=0$. ☐

For $w,u\in\mathfrak{H}^0$ we can also set $z=1$ in the Proposition above and obtain the following.

**Corollary 2.12.** *For any $w,u\in\mathfrak{H}^0$ we have*

$$\zeta(w)\zeta(v) = \zeta(w\sqcup\!\sqcup v)\,.$$

*In particular the space $\mathcal{Z}$ is a $\mathbb{Q}$-subalgebra of $\mathbb{R}$ and $\zeta$ is an algebra homomorphism from $\mathfrak{H}^0_{\sqcup\!\sqcup}$ to $\mathcal{Z}$.*

**Proposition 2.13** (Finite double shuffle relations)**.** *For $w,u\in\mathfrak{H}^0$ we have*

$$\zeta(w\sqcup\!\sqcup u - w*u) = 0\,.$$

But it is also clear, that these do not give all linear relations among multiple zeta values. For example the relation $\zeta(3)=\zeta(2,1)$ is not a consequence of the above Proposition. Counting the finite double shuffle relations, we get the following table, which comes from the survey article [Tan]. In this article, you can also find the numbers of other families of relations, such as the duality relation.

| weight $k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| # all conjectured relations | 1 | 3 | 6 | 14 | 29 | 60 | 123 | 249 | 503 | 1012 |
| # finite double shuffle relations | 0 | 1 | 2 | 7 | 16 | 40 | 92 | 200 | 429 | 902 |

## 2.3 Duality relation

We now give a direct consequence of the iterated integral expression. Making the change of variables $s_j = 1 - t_{k-j+1}$ in the iterated integral expression gives a linear relation among multiple zeta values, which is called the duality relation. For example if $k = 3$ we can make the change of variables $s_1 = 1 - t_3$, $s_2 = 1 - t_2$, $s_3 = 1 - t_1$ in the following iterated integral

$$\begin{aligned}\zeta(3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_3}{1-t_3} &= \int_1^0 \frac{-ds_3}{1-s_3} \int_1^{s_3} \frac{-ds_2}{1-s_2} \int_1^{s_2} \frac{-ds_1}{s_1} \\ &= \int_0^1 \frac{ds_1}{s_1} \int_0^{s_1} \frac{ds_2}{1-s_2} \int_0^{s_2} \frac{ds_3}{1-s_3} = \zeta(2,1)\,.\end{aligned} \tag{2.4}$$

from which we again get the relation $\zeta(3) = \zeta(2,1)$ in Proposition 1.9. This change of variables can be described nicely in terms of an anti-automorphism on the space $\mathfrak{H}$. For this we denote by $\tau$ the anti-automorphism of $\mathfrak{H}$ which interchanges $x$ and $y$. Here we view $\mathfrak{H}$, and all its subspaces, as $\mathbb{Q}$-algebras where the product is given by the usual non-commutative product in $\mathbb{Q}\langle x, y\rangle$. That $\tau$ is an anti-automorphism just means that $\tau(uw) = \tau(w)\tau(u)$ for $u, w \in \mathfrak{H}$ and $\tau(1) = 1$. For example if $w = z_3 = xxy$, then

$$\tau(z_3) = \tau(xxy) = \tau(y)\tau(xx) = \tau(y)\tau(x)\tau(x) = xyy = z_2 z_1\,.$$

Notice that $\tau(\mathfrak{H}^0) \subset \mathfrak{H}^0$, since any non-empty word $w \in \mathfrak{H}^0$ is of the form $w = xuy$ for some $u \in \mathfrak{H}$ and therefore $\tau(w) = \tau(xuy) = \tau(y)\tau(u)\tau(x) = x\tau(u)y \in \mathfrak{H}^0$. Further notice that $\tau$ is an involution, i.e. $\tau^2 = \mathrm{id}_{\mathfrak{H}}$ and $\tau(\mathfrak{H}^0) = \mathfrak{H}^0$.

**Proposition 2.14** (Duality relation)**.** *For all $w \in \mathfrak{H}^0$ we have*

$$\zeta(\tau(w)) = \zeta(w)\,.$$

*Proof.* This is just a generalization of the variable change $s_j = 1 - t_{k-j+1}$ in the iterated integral expression in Corollary 2.10 similar to (2.4). Interchanging $x$ and $y$ corresponds to $\omega_a(t_{k-j+1}) = -\omega_{\tau(a)}(1 - s_j)$ for $a \in \{x, y\}$. The property of $\tau$ being an anti-automorphism corresponds to changing the order/directions of the integrals, which also gets rid of the minus signs. □

A few explicit examples of the duality relations are given by the following Corollary, which both can be seen as a generalization of the formula $\zeta(3) = \zeta(2,1)$. Here we use the common notation $\{k_1, \dots, k_r\}^n = \underbrace{k_1, \dots, k_r \dots k_1, \dots, k_r}_{rn}$ for $n$ copies of the string $k_1, \dots, k_r$.

**Corollary 2.15.** *i) For all $k \geq 3$ we have*

$$\zeta(k) = \zeta(2, \underbrace{1, \dots, 1}_{k-2}) = \zeta(2, \{1\}^{k-2})\,.$$

*ii) For all $n \geq 1$ we have*

$$\zeta(\{2,1\}^n) = \zeta(\{3\}^n)\,.$$

*Proof.* Both statements are immediate consequences of the duality relations, since $\tau(z_k) = \tau(x^{k-1}y) = xy^{k-1} = z_2 z_1 \cdots z_1$ and $\tau((z_2 z_1)^n) = \tau(z_2 z_1)^n = z_3^n$. □

Until here in lecture 6 (3rd June, 2022)

## 2.4 Regularizations & Extended double shuffle relations

We now want to make sense of multiple zeta values for non-admissible, i.e. we want to extend the map $\zeta$ from $\mathfrak{H}^0$ to $\mathfrak{H}^1$. For this we need the following result.

**Proposition 2.16.** *We have*

*i)* $\mathfrak{H}^1_{ш} = \mathfrak{H}^0_{ш}[y]$ *and* $\mathfrak{H}_{ш} = \mathfrak{H}^1_{ш}[x] = \mathfrak{H}^0_{ш}[x, y]$.

*ii)* $\mathfrak{H}^1_* = \mathfrak{H}^0_*[z_1]$.

*Proof.* The second part is Exercise 10 and the first part can be found in [B6]. □

*Example* 2.17. As an example of Proposition 2.16 we give the following expressions of $z_1 z_1 z_2$ as a polynomial in $z_1 = y$ having coefficients in $\mathfrak{H}^0$ with respect to the products ш and $*$

$$z_1 z_1 z_2 = \frac{1}{2} z_2 ш z_1^{ш 2} - 2 z_2 z_1 ш z_1 + 3 z_2 z_1 z_1 \,,$$
$$z_1 z_1 z_2 = \frac{1}{2} z_2 * z_1^{*2} - (z_2 z_1 + z_3) * z_1 + \left( z_2 z_1 z_1 + z_3 z_1 + \frac{1}{2} z_4 \right) .$$

As a consequence of Proposition 2.16) we have for $\bullet \in \{ш, *\}$ isomorphism of $\mathbb{Q}$-algebras

$$\mathrm{reg}^T_\bullet : \mathfrak{H}^1_\bullet \to \mathfrak{H}^0_\bullet[T] \,,$$

which send an element $w = \sum_{j=0}^m w_j \bullet z_1^{\bullet m}$ with $w_j \in \mathfrak{H}^0$ to $\mathrm{reg}^T_\bullet(w) = \sum_{j=0}^m w_j T^m$. This enables us to extend the algebra homomorphism $\zeta : \mathfrak{H}^0_\bullet \to \mathcal{Z}$ to an algebra homomorphism $\zeta^\bullet : \mathfrak{H}^1_\bullet \to \mathcal{Z}[T]$ by extending $\zeta$ to $\mathfrak{H}^0_\bullet[T]$ and setting $\zeta^\bullet = \zeta \circ \mathrm{reg}^T_\bullet$, i.e. we have the following commutative diagram of $\mathbb{Q}$-algebra homomorphism

$$\begin{array}{ccc} \mathfrak{H}^1_\bullet & \xrightarrow{\mathrm{reg}^T_\bullet} & \mathfrak{H}^0_\bullet[T] \\ & \searrow^{\zeta^\bullet} & \downarrow \zeta \\ & & \mathcal{Z}[T] \end{array}$$

**Definition 2.18.** *Let* $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}_{\geq 1}$ *be any index.*

*i) We define the* **shuffle regularized multiple zeta value**

$$\zeta^{ш}(\mathbf{k}; T) = \zeta^{ш}(k_1, \ldots, k_r; T) := \zeta^{ш}(z_\mathbf{k}) \in \mathcal{Z}[T]$$

*In the case* $T = 0$ *we just write* $\zeta^{ш}(\mathbf{k}) = \zeta^{ш}(\mathbf{k}; 0)$.[3]

*ii) We define the* **stuffle regularized multiple zeta value**

$$\zeta^*(\mathbf{k}; T) = \zeta^*(k_1, \ldots, k_r; T) := \zeta^*(z_\mathbf{k}) \in \mathcal{Z}[T]$$

*In the case* $T = 0$ *we just write* $\zeta^*(\mathbf{k}) = \zeta^*(\mathbf{k}; 0)$.

[3] In the literature often "shuffle/shuffle regularized multiple zeta value" refers to the $T = 0$ case.

From Example 2.17 we obtain

$$\begin{aligned}\zeta^{ш}(1,1,2;T) &= \frac{1}{2}\zeta(2)T^2 - 2\zeta(2,1)T + 3\zeta(2,1,1)\,,\\ \zeta^{*}(1,1,2;T) &= \frac{1}{2}\zeta(2)T^2 - (\zeta(2,1)+\zeta(3))T + \zeta(2,1,1) + \zeta(3,1) + \frac{1}{2}\zeta(4)\,.\end{aligned}$$

Even though the coefficients of $T^2$ and $T$ are the same (because we know $\zeta(2,1)=\zeta(3)$), the constant terms differ. In general $\zeta^{ш}(\mathbf{k};T)$ and $\zeta^{ш}(\mathbf{k};T)$ are different if $\mathbf{k}$ is non-admissible. We will present the exact relationship between these two regularizations as it was done in [IKZ]. For this first consider the following series

$$\begin{aligned}A(u) &= \exp\left(\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\zeta(n)u^n\right)\\ &= 1 + \frac{\zeta(2)}{2}u^2 - \frac{\zeta(3)}{3}u^3 + \left(\frac{\zeta(4)}{4} + \frac{\zeta(2)^2}{8}\right)u^4 - \left(\frac{\zeta(5)}{5} + \frac{\zeta(2)\zeta(3)}{6}\right)u^5 + \dots\\ &=: \sum_{k\geq 0}\gamma_k\, u^k\,.\end{aligned}$$

Here the $\gamma_k \in \mathbb{Q}[\zeta(j) \mid j \geq 2]$ are polynomials of single zeta values which, considered as multiple zeta values, have homogeneous weight $k$. Using this we define the $\mathbb{R}$-linear map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ by

$$\rho(e^{Tu}) := A(u)e^{Tu}\,. \tag{2.5}$$

Notice that this defines the linear map $\rho$ uniquely by comparing the coefficients of $u^m$ on both sides. Since $\rho$ is linear, we get

$$\begin{aligned}\rho(e^{Tu}) &= \rho(1) + \rho(T)u + \frac{1}{2}\rho(T^2)u^2 + \frac{1}{3!}\rho(T^3)u^3 + \dots\\ &:= A(u)e^{Tu} = \left(1 + \gamma_1 u + \gamma_2 u^2 + \dots\right)\left(1 + \frac{1}{2}u^2 + \frac{1}{3!}u^3 + \dots\right)\\ &= 1 + Tu + \frac{1}{2}\left(T^2 + \zeta(2)\right)u^2 + \frac{1}{3!}\left(T^3 + 3\zeta(2)T - 2\zeta(3)\right)u^3\,,\end{aligned}$$

and therefore we obtain for $m \geq 0$ the explicit formula

$$\rho\left(T^m\right) = m!\sum_{k=0}^{m}\gamma_k\frac{T^{m-k}}{(m-k)!}\,.$$

Also notice that $\rho$ is bijective. For the first values of $m$ we get

$$\begin{aligned}\rho(1) &= 1\,,\\ \rho(T) &= T\,,\\ \rho(T^2) &= T^2 + \zeta(2)\,,\\ \rho(T^3) &= T^3 + 3\zeta(2)T - 2\zeta(3)\,,\\ \rho(T^4) &= T^4 + 6\zeta(2)T^2 - 8\zeta(3)T + 6\zeta(4) + 3\zeta(2)^2\,,\\ \rho(T^5) &= T^5 + 10\zeta(2)T^3 - 20\zeta(3)T^2 + \left(30\zeta(4) + 15\zeta(2)^2\right)T - 24\zeta(5) - 20\zeta(2)\zeta(3)\,.\end{aligned}$$

The stuffle regularized multiple zeta values are elements in $\mathbb{R}[T]$ and for example we saw that

$$\zeta^*(1,1,2;T) = \frac{1}{2}\zeta(2)T^2 - (\zeta(2,1)+\zeta(3))T + \zeta(2,1,1) + \zeta(3,1) + \frac{1}{2}\zeta(4)\,.$$

Applying the linear map $\rho$ to this, we see that we just get an additional contribution of $\frac{1}{2}\zeta(2)^2$, i.e.

$$\rho\left(\zeta^*(1,1,2;T)\right) = \frac{1}{2}\zeta(2)T^2 - (\zeta(2,1)+\zeta(3))T + \zeta(2,1,1) + \zeta(3,1) + \frac{1}{2}\zeta(4) + \frac{1}{2}\zeta(2)^2\,.$$

Using the known relations $\zeta(3) = \zeta(2,1)$ and $\zeta(4) = \zeta(2,1,1)$ (duality), $\zeta(3,1) = \frac{1}{4}\zeta(4)$ (finite double shuffle) and $\zeta(2)^2 = \frac{5}{2}\zeta(4)$ (Euler), we get

$$\begin{aligned}\rho\left(\zeta^*(1,1,2;T)\right) &= \frac{1}{2}\zeta(2)T^2 - 2\zeta(2,1)T + 3\zeta(2,1,1)\\ &= \zeta^{ш}(1,1,2;T)\,,\end{aligned}$$

i.e. the $\rho$ sends the stuffle regularized multiple zeta value to the shuffle regularized multiple zeta value. In general the map $\rho$ has this property and we have the following

**Theorem 2.19** ([IKZ]). *For all* $\mathbf{k} \in \mathbb{Z}^r_{\geq 1}$ *we have*

$$\zeta^{ш}(\mathbf{k};T) = \rho\left(\zeta^*(\mathbf{k};T)\right)\,.$$

*Or equivalently, when viewed as maps from* $\mathfrak{H}^1$ *to* $\mathbb{R}[T]$*, we have* $\zeta^{ш} = \rho \circ \zeta^*$*.*

As a consequence of Theorem 2.19 and as an extension of the finite double shuffle relations (Proposition 2.13) we will now present a family of linear relations, which give conjecturally all linear relations among multiple zeta values. We define for $w, u \in \mathfrak{H}^1$ the element

$$\mathrm{ds}(w,u) := w ш u - w * u \in \mathfrak{H}^1.$$

The statement of Proposition 2.13 then was, that $\mathrm{ds}(w,u) \in \ker\zeta$ if $w,u \in \mathfrak{H}^0$. The extended version states, that one of the words $w$ and $u$ is allowed to be in $\mathfrak{H}^1$. In this case $\mathrm{ds}(w,u)$ is not necessary in $\mathfrak{H}^0$ anymore, but after projecting to $\mathfrak{H}^0$ by the map $\mathrm{reg}^T_\bullet : \mathfrak{H}^1_\bullet \to \mathfrak{H}^0_\bullet[T]$ and then comparing the coefficients of $T$ (or setting $T=0$, for which we write $\mathrm{reg}_\bullet := \mathrm{reg}^0_\bullet$) one still obtains a relation among multiple zeta values. In other words $\mathrm{ds}(w,u)$ is in the kernel of the regularized multiple zeta value maps.

**Theorem 2.20** (Extended double shuffle relations). *For* $w \in \mathfrak{H}^1$*,* $u \in \mathfrak{H}^0$ *and* $\bullet \in \{ш, *\}$ *we have*

$$\zeta^\bullet(w ш u - w * u;T) = 0\,,$$

*i.e. in particular* $\mathrm{reg}_\bullet\left(\mathrm{ds}(w,u)\right) \in \ker\zeta$.

*Proof.* By Theorem 2.19 we have for all $w \in \mathfrak{H}^1$

$$\zeta^{ш}(w;T) = \rho\left(\zeta^*(w;T)\right)\,.$$

Multiplying both sides with $\zeta^{ш}(u) = \zeta^*(u) = \zeta(u) \in \mathbb{R}$, for $u \in \mathfrak{H}^0$, we get by the $\mathbb{R}$-linearity of $\rho$

$$\begin{aligned}\zeta^{ш}(w ш u;T) &= \rho\left(\zeta^*(w * u;T)\right)\\ &= \zeta^{ш}(w * u;T)\,.\end{aligned}$$

This gives $\zeta^{ш}(w ш u - w * u;T) = 0$ and $\zeta^*(w ш u - w * u;T) = 0$ by applying the inverse of $\rho$. ☐

**Conjecture 2.21.** *The kernel of $\zeta : \mathfrak{H}^0 \to \mathcal{Z}$ is given by*

$$\begin{aligned}\ker\zeta &= \left\langle \operatorname{reg}_{ш}(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}} \\ &= \left\langle \operatorname{reg}_{*}(w ш u - w * u) \mid w \in \mathfrak{H}^1,\, u \in \mathfrak{H}^0 \right\rangle_{\mathbb{Q}},\end{aligned}$$

*i.e. the extended double shuffle relations give all $\mathbb{Q}$-linear relations among multiple zeta values.*

Since it is expected that the extended double shuffle relations give all relations among multiple zeta values, one could obtain upper bounds for the dimension of $\mathcal{Z}_k$, i.e., an alternative proof of Theorem 1.16, by counting these relations. This is still an open problem.

## 2.5 Double shuffle relations for $q$-analogues

In this section, we introduce the algebraic setup and the analogues of the double shuffle relations for $q$-analogues of multiple zeta values. This algebraic setup was, with a slightly different notation, introduced by Takeyama in [Tak] for an extended version of the Bradley-Zhao multiple zeta values. For a good overview how these relate to other models of $q$-analogues we refer to the overview article [Bri]. We will use mainly the notations from [Tak2] together with some self-made modification.

We saw that there is an easy description for the analogue of the harmonic product for the $q$-analogues $\zeta_q$. In order to give an analogue of the shuffle product, one needs to allow some kind of extended versions of $q$-analogues. Recall that for $k \geq 1$ we considered the term $\frac{q^{(k-1)m}}{[m]_q^k}$ in the definition of $\zeta_q$, in particular for $k = 1$ the term $\frac{1}{[m]_q}$ appeared. We will extend this now by introducing a second term for the weight 1 index, which will correspond to $\frac{q^m}{[m]_q}$. For this we define $\overline{\mathbb{N}} = \{\bar{1}\} \cup \mathbb{Z}_{\geq 1}$ and then define for $k \in \overline{\mathbb{N}}$ and $m \geq 1$ the following $q$-series

$$f_k(m) = \begin{cases} \frac{q^m}{[m]_q}, & \text{if } k = \bar{1} \\ \frac{q^{(k-1)m}}{[m]_q^k}, & \text{if } k \in \mathbb{Z}_{\geq 1} \end{cases}. \tag{2.6}$$

With this we extend the definition of $\zeta_q$ in the following way.

**Definition 2.22.** *(i) For $\mathbf{k} = (k_1, \ldots, k_r) \in \overline{\mathbb{N}}^r$ with $k_1 \neq 1$ define the (extended) q-multiple zeta value*

$$\zeta_q(\mathbf{k}) = \zeta_q(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} f_{k_j}(m_j) \in \mathbb{Q}[[q]].$$

*(ii) For any $\mathbf{k} = (k_1, \ldots, k_r) \in \overline{\mathbb{N}}^r$ define for q-multiple polylogarithm by*

$$\mathrm{Li}^q_{\mathbf{k}}(z) = \mathrm{Li}^q_{k_1,\ldots,k_r}(z) = \sum_{m_1 > \cdots > m_r > 0} z^{m_1} \prod_{j=1}^{r} f_{k_j}(m_j) \in \mathbb{Q}[[q]][[t]].$$

Notice that we view both objects as formal power series, but both can also be viewed as complex functions in $q, z \in \mathbb{C}$ with $|q|, |z| < 1$. In the case $k_1 \neq 1$ we can also consider the limit $\lim_{z\to 1} \mathrm{Li}^q_{\mathbf{k}}(z) = \zeta_q(\mathbf{k})$ and in the case $k_1 \neq \bar{1}, 1$ we have $\lim_{q \to 1} \zeta_q(\mathbf{k}) = \zeta(\mathbf{k})$. Moreover for any $\mathbf{k} = (k_1, \ldots, k_r) \in \overline{\mathbb{N}}^r$ we have $\lim_{q\to 1} \mathrm{Li}^q_{\mathbf{k}} = \mathrm{Li}_{\mathbf{k}}$.

The goal is to introduce an algebraic setup, which considers several different $q$-analogue models at the same time. We saw that the additional index $\bar{1}$ will corresponds to a factor $\frac{q^m}{[m]_q}$ in the definition of $\zeta_q$. We want to generalize this even more in our setup and allow indices $k, \bar{k}$ and $\hat{k}$. For each of these we will introduce a letter $e_k, e_{\bar{k}}$ and $e_{\hat{k}}$. The rough idea is that we have the following correspondences, which indicates the relationship of letters to their corresponding factor in the definition of $\zeta_q$

$$\begin{aligned} e_k &\longleftrightarrow \frac{q^{(k-1)m}}{[m]_q^k} \\ e_{\hat{k}} &\longleftrightarrow \frac{q^{km}}{[m]_q^k} \\ e_{\bar{k}} &\longleftrightarrow \frac{q^m}{[m]_q^k}\,. \end{aligned} \tag{2.7}$$

Notice that all these factors on the right-side evaluate to $\frac{1}{m^k}$ as $q \to 1$. Since we have

$$\frac{q^{km}}{[m]_q^k} = (-1)^{k-1}(1-q)^{k-1}\frac{q^m}{[m]_q} + \sum_{j=2}^{k}(-1)^{k-j}(1-q)^{k-j}\frac{q^{(j-1)m}}{[m]_q^j}\,, \qquad (k \geq 1) \tag{2.8}$$

$$\frac{q^m}{[m]_q^k} = \sum_{j=2}^{k}\binom{k-2}{j-2}(1-q)^{k-j}\frac{q^{(j-1}m}{[m]_q^j}\,, \qquad (k \geq 2)$$

it is sufficient to just introduce $e_{\bar{1}}$ and $e_k$ for $k \geq 1$ in our setup, and then define $e_{\hat{k}}$ and $e_{\bar{k}}$ in terms of $e_{\bar{1}}$ and $e_k$ by using the formulas (4.8). In our algebraic setup the term $(1-q)$ will correspond to a formal variable $\hbar$ and instead of working with $\mathbb{Q}$-vector spaces we will work with $\mathcal{C}$-modules, where

$$\mathcal{C} = \mathbb{Q}[\hbar, \hbar^{-1}]\,.$$

**Definition 2.23.** *(i) We define $\widehat{\mathfrak{H}} = \mathcal{C}\langle a, b\rangle$ and its two $\mathcal{C}$-submodules*

$$\widehat{\mathfrak{H}}^0 = \mathcal{C} + a\widehat{\mathfrak{H}}b \quad \subset \quad \widehat{\mathfrak{H}}^1 = \mathcal{C} + \widehat{\mathfrak{H}}b \quad \subset \quad \widehat{\mathfrak{H}}\,.$$

*(ii) For $k \geq 1$ we write $e_k = a^{k-1}(a+\hbar)b$ and $e_{\bar{1}} = ab$. With this we have $b = \frac{1}{\hbar}(e_1 - e_{\bar{1}})$ and*

$$\begin{aligned} \widehat{\mathfrak{H}}^1 &= \mathcal{C}\langle e_{\bar{1}}, e_1, e_2, e_3, \dots\rangle, \\ \widehat{\mathfrak{H}}^0 &= \mathcal{C} + \langle e_{k_1}\dots e_{k_r} \mid r \geq 1, k_1, \dots, k_r \in \overline{\mathbb{N}}, k_1 \neq 1\rangle_{\mathcal{C}}\,. \end{aligned}$$

*(iii) We set $e_{\hat{1}} = e_{\bar{1}} = ab$ and for $k \geq 2$ define, inspired by (2.7) and (4.8),*

$$\begin{aligned} e_{\hat{k}} &:= (-1)^{k-1}\hbar^{k-1}e_{\bar{1}} + \sum_{j=2}^{k}(-1)^{k-j}\hbar^{k-j}e_j = a^k b\,, \\ e_{\bar{k}} &:= \sum_{j=2}^{k}\binom{k-2}{j-2}\hbar^{k-j}e_j\,. \end{aligned} \tag{2.9}$$

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We define a $\mathcal{C}$-module structure on $\mathbb{Q}[[q]]$ and $\mathbb{Q}[[q]][[z]]$ by defining the multiplication of $\hbar$ to be the multiplication with $(1-q)$. Since the words in $\widehat{\mathfrak{H}}^0$ correspond exactly to those indices for which we defined $\zeta_q$, we can define the $\mathcal{C}$-linear map

$$\begin{aligned}\zeta_q\colon \widehat{\mathfrak{H}}^0 &\longrightarrow \mathbb{Q}[[q]]\\ e_{\mathbf{k}} := e_{k_1}\dots e_{k_r} &\longmapsto \zeta_q(\mathbf{k})\,,\end{aligned}$$

where $\mathbf{k}=(k_1,\dots,k_r)\in\overline{\mathbb{N}}^r$. Similarly we can define the $\mathcal{C}$-linear map

$$\begin{aligned}\mathrm{Li}^q\colon \widehat{\mathfrak{H}}^1 &\longrightarrow \mathbb{Q}[[q]][[z]]\\ e_{\mathbf{k}} &\longmapsto \mathrm{Li}^q_{\mathbf{k}}(z)\,.\end{aligned}$$

Using (4.7) we extend $\zeta_q(\mathbf{k})$ and $\mathrm{Li}^q_{\mathbf{k}}$ to indices $\mathbf{k}=(k_1,\dots,k_r)\in\{k,\bar{k},\widehat{k}\mid k\in\mathbb{Z}_{\geq1}\}$ by setting $\zeta_q(k_1,\dots,k_r):=\zeta_q(e_{k_1}\dots e_{k_r})$ and $\mathrm{Li}^q_{k_1,\dots,k_r}(z):=\mathrm{Li}^q_{e_{k_1}\dots e_{k_r}}$. For example, using (2.7) and (4.8) we get

$$\zeta_q(\widehat{9},3,\bar{4}) = \sum_{m_1>m_2>m_3>0}\frac{q^{9m_1}}{[m_1]_q^9}\frac{q^{2m_2}}{[m_2]_q^3}\frac{q^{m_3}}{[m_3]_q^4}\,.$$

One can embed the classical $\mathfrak{H}$ into $\widehat{\mathfrak{H}}$ in various different ways. For example, the embedding $\iota: x\mapsto x$, $\iota: y\mapsto (a+\hbar)b$ yields $\iota(z_k)=e_k$. Another variant would be $\widehat{\iota}\colon x\mapsto x$, $\widehat{\iota}\colon y\mapsto ab$ yielding $\widehat{\iota}(z_k)=e_{\widehat{k}}$. To extend the $q$-stuffle product $*_q$, defined in Example 2.3 (iii) on the space $\mathfrak{H}^1$, we first observe that the functions $f_k$ defined in (4.6) satisfy for $k,k_1,k_2\in\mathbb{Z}_{\geq1}$

$$f_{\bar{1}}^2 = f_2-(1-q)f_{\bar{1}},\qquad f_{\bar{1}}f_k=f_{k+1},\qquad f_{k_1}f_{k_2}=f_{k_1+k_2}+(1-q)f_{k_1+k_2-1}\,.$$

Motivated by this we define $*_q=*_\diamond$ to be the quasi-shuffle product[4] on $\widehat{\mathfrak{H}}^1=\mathcal{C}\langle\widehat{A}\rangle$ defined by the alphabet $\widehat{A}=\{e_{\bar{1}},e_1,e_2,\dots\}$ and the product $\diamond$ on $\mathcal{C}\widehat{A}$ given by

$$e_{\bar{1}}\diamond e_{\bar{1}}=e_2-\hbar e_{\bar{1}},\qquad e_{\bar{1}}\diamond e_k=e_{k+1},\qquad e_{k_1}\diamond e_{k_2}=e_{k_1+k_2}+\hbar e_{k_1+k_2-1}\,.$$

Notice that we have $e_{\widehat{k_1}}\diamond e_{\widehat{k_1}}=e_{\widehat{k_1+k_2}}$ for any $k_1,k_2\geq1$. This gives a commutative $\mathcal{C}$-algebra $\widehat{\mathfrak{H}}^1_{*_q}$. The subspace $\widehat{\mathfrak{H}}^0$ is closed under $*_q$, which gives a subalgebra $\widehat{\mathfrak{H}}^0_{*_q}\subset\widehat{\mathfrak{H}}^1_{*_q}$, and a consequence of Lemma 2.4 we obtain:

**Proposition 2.24.** *The map $\zeta_q:\widehat{\mathfrak{H}}^0_{*_q}\to\mathbb{Q}[[q]]$ is an $\mathcal{C}$-algebra homomorphism.*

Notice that $*_q$ is a natural extension of same-named quasi-shuffle product on $\mathfrak{H}^1$, since $\iota:\mathfrak{H}^1_{*_q}\to\widehat{\mathfrak{H}}^1_{*_q}$ is an injective $\mathcal{C}$-algebra homomorphism.

We now turn our attention to the analogue of the shuffle product for $q$-analogues. For this we define an action of $\widehat{\mathfrak{H}}$ on the $\mathcal{C}$-module $z\mathbb{Q}[[q,z]]$ for an $g\in z\mathbb{Q}[[q,z]]$ by $1g=g$ and

$$\begin{aligned}ag &= (1-q)\sum_{j=1}^{\infty} g_{|z=q^jz}\\ bg &= \frac{z}{1-z}g\,.\end{aligned}$$

[4]Notice that the original definition of quasi-shuffle products is defined over fields and not rings. But all results we use here for quasi-shuffle products also work for algebras over rings.

For any word $w = l_1 \dots l_m \in \widehat{\mathfrak{H}}$ with $l_j \in \{a, b\}$ we then define recursively $wg = (l_1 \dots l_{m_1})(l_m g)$ and extend this $\mathcal{C}$-linearly to all of $\widehat{\mathfrak{H}}$. Notice that this action is just well defined on $z\mathbb{Q}[[q, z]]$ since the action of $a$ would not be well-defined for power series having a constant term with respect to $z$. But we can extend the action to $\mathbb{Q}[[q, z]]$ for elements in $w \in \widehat{\mathfrak{H}}^1$, since $b\mathbb{Q}[[q, z]] \subset z\mathbb{Q}[[q, z]]$.

*Example* 2.25. Consider the element $1 \in \mathbb{Q}[[q, z]]$ and $w = e_{\bar{1}} = ab$, then we have

$$\begin{aligned} e_{\bar{1}}1 = ab1 = a\frac{z}{1-z} &= (1-q)\sum_{j=1}^{\infty} \frac{q^j z}{1-q^j z} = (1-q)\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}(q^j z)^m \\ &= \sum_{m>0} \frac{(1-q)z^m q^m}{(1-q^m)} = \sum_{m>0}\frac{z^m q^m}{[m]_q} = \mathrm{Li}^q_{\bar{1}}(z)\,. \end{aligned}$$

If we want to calculate $e_{\hat{2}}1 = aab1$ we can use above calculations and get

$$e_{\hat{2}}1 = a\,\mathrm{Li}^q_{\bar{1}}(z) = (1-q)\sum_{j=1}^{\infty}\sum_{m>0}\frac{(q^j z)^m q^m}{[m]_q} = \sum_{m>0}\frac{q^{2m} z^m (1-q)}{(1-q^m)[m_q]} = \sum_{m>0}\frac{z^m q^{2m}}{[m]_q^2} = \mathrm{Li}^q_{\hat{2}}(z)\,.$$

In general we obtain the following.

**Proposition 2.26.** *For all $w \in \widehat{\mathfrak{H}}^1$ we have $\mathrm{Li}^q_w = w1$.*

*Proof.* This is Exercise 11. □

Now we observe (Exercise 12) that for $f, g \in z\mathbb{Q}[[q, z]]$ we have

$$\begin{aligned} (af)(ag) &= a((af)g + f(ag) + \hbar fg), \\ (bf)g &= f(bg) = b(fg)\,. \end{aligned} \tag{2.10}$$

Motivated by Proposition 2.26 and (2.10) we give the following definition.

**Definition 2.27.** *We define the q-shuffle product $ш_q$ on $\widehat{\mathfrak{H}}$ as the $\mathcal{C}$-bilinear product, which satisfies $1 ш_q w = w ш_q 1 = w$ for any $w \in \widehat{\mathfrak{H}}$ and*

$$\begin{aligned} aw ш_q av &= a((aw) ш_q v + w ш_q (av) + \hbar w ш_q v)\,, \\ (bw) ш_q v &= w ш_q (bw) = b(w ш_q v) \end{aligned}$$

*for words $w, v \in \widehat{\mathfrak{H}}$.*

*Example* 2.28. As an example we calculate

$$e_{\bar{1}} ш_q e_{\bar{1}} = ab ш_q ab = 2abab + \hbar abb = 2e_{\bar{1}}e_{\bar{1}} + \hbar e_{\bar{1}} b = 2e_{\bar{1}}e_{\bar{1}} + e_{\bar{1}}e_1 - e_{\bar{1}}e_{\bar{1}}\,.$$

and ( {♨ **Todo:** write out the details ♨})

$$\begin{aligned} e_{\hat{2}} ш_q e_{\hat{2}} &= 4e_{\hat{3}}e_{\hat{1}} + 2e_{\hat{2}}e_{\hat{2}} + \hbar(2e_{\hat{3}}b + 4e_{\hat{2}}e_{\hat{1}}) + \hbar^2 e_{\hat{2}} b \\ &= 4e_{\hat{3}}e_{\hat{1}} + 2e_{\hat{2}}e_{\hat{2}} + 2e_{\hat{3}}e_1 - 2e_{\hat{3}}e_{\bar{1}} + \hbar(e_{\hat{2}}e_1 + 2e_{\hat{2}}e_{\bar{1}})\,. \end{aligned}$$

Compare this to the classical shuffle product $z_2 ш z_2 = 4z_3z_2 + 2z_2z_2$.

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**Proposition 2.29.** *(i) The space $\widehat{\mathfrak{H}}_{\sqcup\!\sqcup_q} = (\widehat{\mathfrak{H}}, \sqcup\!\sqcup_q)$ is a commutative $\mathcal{C}$-algebra.*

*(ii) The spaces $\widehat{\mathfrak{H}}^1_{\sqcup\!\sqcup_q} = (\widehat{\mathfrak{H}}^1, \sqcup\!\sqcup_q)$ and $\widehat{\mathfrak{H}}^0_{\sqcup\!\sqcup_q} = (\widehat{\mathfrak{H}}^0, \sqcup\!\sqcup_q)$ are subalgebras of $\widehat{\mathfrak{H}}_{\sqcup\!\sqcup_q}$.*

*Proof.* The proofs are similar to the case of quasi-shuffle products. □

*Remark* 2.30. In general one can check that $\sqcup\!\sqcup_q$ can be seen as the $q$-analogue of the shuffle product $\sqcup\!\sqcup$, since the $q \to 1$ in our algebraic setup corresponds to setting $\hbar = 0$. More precisely consider the following projection

$$\tilde{\mathfrak{H}}^1 := \mathbb{Q}[\hbar]\langle e_{\bar{1}}, e_1, e_2, \dots \rangle \underset{\hbar=0}{\longrightarrow} \mathbb{Q} + \mathbb{Q}\langle a, ab \rangle ab \underset{\substack{ab \mapsto y \\ a \mapsto x}}{\longrightarrow} \mathfrak{H}^1 = \mathbb{Q} + \mathbb{Q}\langle x, y \rangle y = \mathbb{Q}\langle z_1, z_2, \dots \rangle \,.$$

The space $\tilde{\mathfrak{H}}^1 \subset \widehat{\mathfrak{H}}^1$ is closed under $\sqcup\!\sqcup_q$ and the above map gives an algebra homomorphism from $\tilde{\mathfrak{H}}^1_{\sqcup\!\sqcup_q}$ to $\mathfrak{H}^1_{\sqcup\!\sqcup}$, which can be checked directly by the definition of $\sqcup\!\sqcup_q$ after setting $\hbar = 0$ and observing that in this case $e_{\hat{k}} = e_{\bar{k}} = e_k = a^k b$.

**Proposition 2.31.** *The maps $\mathrm{Li}^q : \widehat{\mathfrak{H}}^1_{\sqcup\!\sqcup_q} \to \mathbb{Q}[[q,z]]$ and $\zeta_q : \widehat{\mathfrak{H}}^0_{\sqcup\!\sqcup_q} \to \mathbb{Q}[[q]]$ are $\mathcal{C}$-algebra homomorphisms.*

*Proof.* To show that $\mathrm{Li}_q$ is an $\mathcal{C}$-algebra homomorphism we need to check that for $w, v \in \widehat{\mathfrak{H}}^1$ we have $\mathrm{Li}^q(w)\,\mathrm{Li}_q(v) = \mathrm{Li}^q(w \sqcup\!\sqcup_q v)$. This can be done similar as in Proposition 2.11, i.e. by induction on $\ell(w) + \ell(v)$ together with Proposition 2.26 and (2.10). The second statement follows by taking the limit $z \to 1$. □

As an analogue of the finite double shuffle relations we obtain the following.

**Corollary 2.32.** *For $w, v \in \widehat{\mathfrak{H}}^0$ we have*

$$\zeta_q(w \sqcup\!\sqcup_q v - w *_q v) = 0 \,.$$

*Examples* 2.33. (i) For $w = v = e_{\bar{1}}$ we get

$$e_{\bar{1}} \sqcup\!\sqcup_q e_{\bar{1}} - e_{\bar{1}} *_q e_{\bar{1}} = 2e_{\bar{1}}e_{\bar{1}} + \hbar \underbrace{e_{\bar{1}}b}_{abb} - \left( 2e_{\bar{1}}e_{\bar{1}} + \underbrace{e_2 - \hbar e_{\bar{1}}}_{aab} \right) = \hbar abb - aab = e_{\bar{1}}e_1 - e_{\bar{1}}e_{\bar{1}} - e_2 + \hbar e_{\bar{1}} \,,$$

which gives the relation

$$0 = (1-q)\zeta_q(abb) - \zeta_q(aab) = \zeta_q(\bar{1}, 1) - \zeta_q(\bar{1}, \bar{1}) - \zeta_q(2) + (1-q)\zeta_q(\bar{1}) \,.$$

(ii) Exercise 13: $w = e_{\bar{1}}$, $v = e_{\hat{2}}$.

(iii) For $w = v = e_{\hat{2}} = aab$ we get

$$e_{\hat{2}} \sqcup\!\sqcup_q e_{\hat{2}} - e_{\hat{2}} *_q e_{\hat{2}} = 4e_{\hat{3}}e_{\hat{1}} + 2e_{\hat{2}}e_{\hat{2}} + 2e_{\hat{3}}e_1 - 2e_{\hat{3}}e_{\bar{1}} + \hbar(e_{\hat{2}}e_1 + e_{\hat{2}}e_{\bar{1}}) - (2e_{\hat{2}}e_{\hat{2}} + e_{\hat{4}}) \,.$$

For some $w, v \in \widehat{\mathfrak{H}}^0$ (but not all!) the limit $q \to 1$ of the individual terms in $\zeta_q(w \sqcup\!\sqcup_q v - w *_q v)$, exist. Notice that the relations among $\zeta$ obtained from this are more than just the finite double shuffle relations in Proposition 2.13. In particular, we obtain the relation $\zeta(3) = \zeta(2,1)$.

It turns out that the relations in Corollary 2.34 do not give all $\mathcal{C}$-linear relations among $\zeta_q$. It was first observed in [Tak], that there is another family of linear relations among $\zeta_q$, called the resummation duality. The same family of linear relations was also discussed for other $q$-analogues, e.g. in [B4] these are called partition relations and in [Bri] the name $SZ$-duality is used. In fact, from an algebraic point of view, these dualities are similar defined as for multiple zeta values. Example 2.33 (i) shows that $\hbar abb - aab \in \ker \zeta_q$. In general, we define the anti-automorphism $\sigma : \widehat{\mathfrak{H}} \to \widehat{\mathfrak{H}}$ by $\sigma(1) = 1$, $\sigma(a) = \hbar b$ and $\sigma(b) = \hbar^{-1} a$, e.g. $\sigma(aab) = \hbar abb$. With this we have the following:

**Proposition 2.34.** *For all $w \in \widehat{\mathfrak{H}}^0$ we have*

$$\zeta_q(w) = \zeta_q(\sigma(w))\,. \tag{2.11}$$

*Proof.* This is a consequence of Theorem 4 in [Tak]. □

Even though the definition of $\sigma$ and $\tau$ are quite similar, the relations obtained are essentially different. Finally we mention the following conjecture, which can be seen as the $q$-analogue version of Conjecture 2.21, which was first indirectly mentioned in [Tak] and then later verified for higher weights by the lecturer.

**Conjecture 2.35.** *All $\mathcal{C}$-linear relations among $\zeta_q$ can be obtained by combining Corollary 2.34 and Proposition 2.34.*

Another nice property of the involution $\sigma$ is that it translates the $q$-harmonic product $*_q$ into the $q$-shuffle product $⧢_q$.

**Proposition 2.36.** *For all $w, v \in \widehat{\mathfrak{H}}^0$ we have*

$$w ⧢_q v = \sigma(\sigma(w) *_q \sigma(v))\,.$$

*Proof.* {☕ **Todo:** include references or proof ☕}. □

With this Conjecture 2.35 has the following reinterpretation.

**Conjecture 2.37.** *All algebraic/linear relations among $\zeta_q$ are a consequence of the $q$-harmonic product $*_q$ and the $\sigma$-invariance* (2.11).

# §3 Finite & symmetric multiple zeta values

We will now turn our attention back to finite and symmetric multiple zeta values.

## 3.1 Linear shuffle relations for finite MZV

Using the notation introduced in the previous section, we can view the finite multiple zeta values as a $\mathbb{Q}$-linear map $\zeta_{\mathcal{A}} : \mathfrak{H}^1 \to \mathcal{Z}^{\mathcal{A}}$. By Lemma 2.4 we see that $\zeta_{\mathcal{A}}$ is an algebra homomorphism from $\mathfrak{H}^1_*$ to $\mathcal{Z}^{\mathcal{A}}$, i.e. for any $w, v \in \mathfrak{H}^1$ we have

$$\zeta_{\mathcal{A}}(w)\zeta_{\mathcal{A}}(v) = \zeta_{\mathcal{A}}(w * v)\,.$$

So we see that the product of two finite multiple zeta values can be expressed in the same way as multiple zeta values. For later we saw, that we have also have the shuffle product formula $\zeta(w)\zeta(v) = \zeta(w ⧢ v)$ for $w, v \in \mathfrak{H}^0$. We expect[5] that this formula is not true for finite multiple zeta values. Instead we have the following **linear shuffle relations** for finite multiple zeta values. For this recall that for $w = z_{k_1} \dots z_{k_r} \in \mathfrak{H}^1$ we denote its reverse by $\overline{w} = z_{k_r} \dots z_{k_1}$

[5] Since we can not prove that $\zeta_{\mathcal{A}}(w) \neq 0$ for any non-empty word $w \in \mathfrak{H}^1$, we can not say that the shuffle product formula is not satisfied.

**Theorem 3.1.** *For all $w, v \in \mathfrak{H}^1$ we have*

$$\zeta_{\mathcal{A}}(w \sqcup\!\sqcup\, v) = (-1)^{\mathrm{wt}(w)} \zeta_{\mathcal{A}}(\overline{w}v)\,,$$

*where $\overline{w}v$ denotes the concatenation of the words $\overline{w}$ and $v$.*

*Proof.* For $n \geq 0$ we write $c_n(\sum_{n\geq 0} a_n z^n) = a_n$ to denote the $n$-th coefficient of a power series in $z$. Then we have for an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{\geq 1}$ and a prime $p$

$$H_{p-1}(\mathbf{k}) = \sum_{p>m_1>\dots>m_r>0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} = \sum_{p>m>0} c_m \left( \underbrace{\sum_{m_1>\dots>m_r>0} \frac{z^{m_1}}{m_1^{k_1} \dots m_r^{k_r}}}_{\mathrm{Li}_{\mathbf{k}}(z)} \right).$$

In general, we have for $w \in \mathfrak{H}^1$

$$H_{p-1}(w) = \sum_{p>m>0} c_m(\mathrm{Li}_w(z))\,.$$

Using Proposition 2.11 ($\mathrm{Li}_{w \sqcup\!\sqcup v}(z) = \mathrm{Li}_w(z)\,\mathrm{Li}_v(z)$) we get for $w = z_{k_1} \dots z_{k_r}$ and $v = z_{l_1} \dots z_{l_s}$

$$\begin{aligned} H_{p-1}(w \sqcup\!\sqcup\, v) &= \sum_{p>m>0} c_m( \overbrace{\mathrm{Li}_{w\sqcup\!\sqcup v}(z)}^{\mathrm{Li}_w(z)\,\mathrm{Li}_v(z)} ) = \sum_{p>m>0} \sum_{\substack{p>i,j>0\\ i+j=m}} c_i(\mathrm{Li}_w(z)) c_j(\mathrm{Li}_v(z)) \\ &= \sum_{\substack{p>i,j>0\\ p>i+j>0}} \sum_{i>m_2>\dots>m_r>0} \frac{1}{i^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \sum_{j>n_2>\dots>n_s>0} \frac{1}{j^{l_1} n_2^{l_2} \cdots n_s^{l_s}}\,. \end{aligned}$$

If we consider this sum modulo $p$, reversing the order of the first sum, and doing the change of variables $i' = p - i$, $m'_t = p - m_t$ $(t = 2, \dots, r)$, we obtain

$$\begin{aligned} H_{p-1}(w \sqcup\!\sqcup\, v) &\equiv \sum_{\substack{p>i,j>0\\ p>i+j>0}} \sum_{i>m_2>\dots>m_r>0} \frac{(-1)^{k_1+\dots+k_r}}{(p-i)^{k_1}(p-m_2)^{k_2} \dots (p-m_r)^{k_r}} \sum_{j>n_2>\dots>n_s>0} \frac{1}{j^{l_1} n_2^{l_2} \dots n_s^{l_s}} \mod p \\ &\equiv \sum_{p>m'_r>\dots>m'_2>i'>j>n_2>\dots>n_s>0} \frac{(-1)^{k_1+\dots+k_r}}{{m'}_r^{k_r} \cdots {m'}_2^{k_2} {i'}^{k_1} j^{l_1} n_2^{l_2} \cdots n_s^{l_s}} \mod p \\ &\equiv (-1)^{\mathrm{wt}(w)} H_{p-1}(\overline{w}v) \mod p, \end{aligned}$$

which implies the statement in the theorem. ☐

**Corollary 3.2.** *For all $w \in \mathfrak{H}^1$ we have $\zeta_{\mathcal{A}}(w) = (-1)^{\mathrm{wt}(w)} \zeta_{\mathcal{A}}(\overline{w})$.*

*Proof.* This is the reversal formula (1.8), but now can also be seen as a special case of above Theorem, by using $v = 1$. ☐

**Conjecture 3.3** (Kaneko-Zagier)**.** *All algebraic and linear relations over $\mathbb{Q}$ among finite multiple zeta values can be deduced from ($w, v \in \mathfrak{H}^1$)*

$$\zeta_{\mathcal{A}}(w * v) = \zeta_{\mathcal{A}}(w)\zeta_{\mathcal{A}}(v), \tag{$*$}$$

$$\zeta_{\mathcal{A}}(w \sqcup\!\sqcup\, v) = (-1)^{\mathrm{wt}(w)} \zeta_{\mathcal{A}}(\overline{w}v) \tag{$\sqcup\!\sqcup$}$$

*In other words: If we define $K$ to be the ideal in $\mathfrak{H}^1_*$ generated by $w ⧢ v - (-1)^{\mathrm{wt}(w)}\overline{w}v$ for all $w, v \in \mathfrak{H}^1$, then we expect that* $\ker(\zeta_{\mathcal{A}}) = K$.

In particular, all relations we proved so far for finite multiple zeta values should be consequenec of $(*)$ and $(⧢)$ (Exercise 14: Show that $\zeta_{\mathcal{A}}(k) = 0$ for all $k \geq 1$). Since $\zeta_{\mathcal{A}}(k) = 0$ the equation $(*)$ reduces in the special case $v = z_k$ to $\zeta_{\mathcal{A}}(w * z_k) = 0$. It seems that this equation is enough to reduce all linear relations together with $(⧢)$:

**Conjecture 3.4** (Kaneko-Zagier)**.** *All $\mathbb{Q}$-linear relations among finite multiple zeta values can be deduces from $(⧢)$ and $\zeta_{\mathcal{A}}(w * z_k) = 0$ for $w \in \mathfrak{H}^1$ and $k \geq 1$.*

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## 3.2 Symmetric MZV

Recall that for an index $\mathbf{k} = (k_1, \dots, k_r)$ and $\bullet \in \{⧢, *\}$ the **$\bullet$-symmetric multiple zeta value** is defined by

$$\zeta^\bullet_{\mathcal{S}}(\mathbf{k}) = \zeta^\bullet_{\mathcal{S}}(k_1, \dots, k_r) = \sum_{j=0}^{r} (-1)^{k_1+\dots+k_j} \zeta^\bullet(k_j, k_{j-1}, \dots, k_1; T)\zeta^\bullet(k_{j+1}, \dots, k_r; T)\,.$$

In this section, we want to sketch the proofs of Proposition 1.29, i.e we want to show the following

(i) We have $\zeta^\bullet_{\mathcal{S}}(\mathbf{k}) \in \mathcal{Z}$ for $\bullet \in \{⧢, *\}$.
(i.e. the $\bullet$-symmetric multiple zeta values are independent of $T$).

(ii) For all indices $\mathbf{k}$ we have

$$\lim_{m\to\infty} S_m(\mathbf{k}) = \zeta^*_{\mathcal{S}}(\mathbf{k})\,.$$

(iii) For all indices $\mathbf{k}$ we have

$$\zeta^*_{\mathcal{S}}(\mathbf{k}) \equiv \zeta^{⧢}_{\mathcal{S}}(\mathbf{k}) \mod \pi^2\mathcal{Z}\,.$$

For this we will make use of the following Lemma.

**Lemma 3.5.** *For an admissible index $\mathbf{k}$ and $\bullet \in \{⧢, *\}$ we define the following generating series*

$$G^\bullet(\mathbf{k}; X, T) := \sum_{j=0}^{\infty} \zeta^\bullet(\{1\}^j, \mathbf{k}; T)X^j$$

*and write in the special case $T = 0$ just $G^\bullet(\mathbf{k}; X) := G^\bullet(\mathbf{k}; X, 0)$.*

*(i) For $\bullet \in \{⧢, *\}$ we have*

$$G^\bullet(\mathbf{k}; X, T) = e^{XT} G^\bullet(\mathbf{k}; X)\,.$$

*(ii) We have*

$$G^*(\mathbf{k}; X) = \Gamma(1 + X)^{-1} e^{-\gamma X} G^{⧢}(\mathbf{k}; X)\,.$$

*Proof.* Statement (i) is given by [IKZ, Proposition 10] ( {♨ **Todo:** Include a proof or sketch♨}. For (ii) recall that we defined the linear map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ in (2.5) by $\rho(e^{Tu}) = A(u)e^{Tu}$, where

$$A(u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n}\zeta(n)u^n\right) = \Gamma(1+u)e^{\gamma u}\,.$$

Here the second equation follows by considering the logarithmic derivative for the Gamma function and $\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log(n)\right)$ denotes Euler's constant. By Theorem 2.19 we have $\zeta^{ш} = \rho\circ\zeta^*$. Applying $\rho$ to (i) with $\bullet = *$ and setting $T = 0$ yields (ii), since $\rho$ is $\mathbb{R}$-linear and $\zeta^*(\{1\}^j, \mathbf{k}, 0) \in \mathbb{R}$. □

For the proof of Proposition 1.29 (i) and (iii) we will consider certain partial sums of the terms appearing in the definition of $\zeta_{\mathcal{S}}$. First notice that the statements in (i) and (iii) are trivial for indices $\mathbf{k} = (k_1, \dots, k_r)$ with $k_1, \dots, k_r \geq 2$, since in this case no regularization is necessary and we have $\zeta_{\mathcal{S}}^{ш}(\mathbf{k}) = \zeta_{\mathcal{S}}^{*}(\mathbf{k}) \in \mathbb{R}$. Now suppose that $\mathbf{k} = (k_1, \dots, k_r) = (\bar{\mathbf{l}}, \{1\}^h, \mathbf{m})$ for $h \geq 1$ and some admissible indices $\mathbf{l}$ and $\mathbf{m}$. Then we claim that the statements in Proposition 1.29 (i) and (iii) are already true for the following partial sum of $\zeta^\bullet(\mathbf{k})$

$$\sum_{i=0}^{h}(-1)^{i+\mathrm{wt}(\mathbf{l})}\zeta^\bullet(\{1\}^i, \mathbf{l}; T)\zeta^\bullet(\{1\}^{h-i}, \mathbf{m}; T)\,. \tag{3.1}$$

Notice that (3.1) is the coefficient of $X^h$ in

$$P^\bullet(\mathbf{l}, \mathbf{m}; X, T) := (-1)^{\mathrm{wt}(\mathbf{l})}G^\bullet(\mathbf{l}; -X, T)G^\bullet(\mathbf{m}; X, T)\,.$$

Now we are ready to give the proofs of Proposition 1.29.

*Proof of Proposition 1.29 (i).* By Lemma 3.5 (i) we have

$$P^\bullet(\mathbf{l}, \mathbf{m}; X, T) = (-1)^{\mathrm{wt}(\mathbf{l})}e^{-XT}G^\bullet(\mathbf{l}; -X)e^{XT}G^\bullet(\mathbf{m}; X) = (-1)^{\mathrm{wt}(\mathbf{l})}G^\bullet(\mathbf{l}; -X)G^\bullet(\mathbf{m}; X)$$

and therefore (3.1) is independent of $T$ and so is $\zeta_{\mathcal{S}}^\bullet(\mathbf{k})$. □

*Proof of Proposition 1.29 (iii).* Since $P$ is independent of $T$ we will just write $P^\bullet(\mathbf{l}, \mathbf{m}; X)$ in the following. By Lemma 3.5 (i),(ii) we obtain

$$P^*(\mathbf{l}, \mathbf{m}; X) = \frac{1}{\Gamma(1+X)\Gamma(1-X)}P^{ш}(\mathbf{l}, \mathbf{m}; X).$$

Using the well-known identity for the sine

$$\frac{1}{\Gamma(1+X)\Gamma(1-X)} = \frac{\sin(\pi X)}{\pi X} = \sum_{n=0}^{\infty}(-1)^n\zeta(\{2\}^n)X^n = \sum_{n=0}^{\infty}(-1)^n\frac{\pi^{2n}X^{2n}}{(2n+1)!} = 1 - \frac{\pi^2}{6}X^2 + \dots$$

we get $P^*(\mathbf{l}, \mathbf{m}; X) - P^{ш}(\mathbf{l}, \mathbf{m}; X) \in \pi^2\mathcal{Z}[[X]]$. This shows that (3.1) for $\bullet = *$ and $\bullet = ш$ are the same modulo $\pi^2\mathcal{Z}$ and therefore the same also holds for $\zeta_{\mathcal{S}}^\bullet(\mathbf{k})$. □

As a direct consequence of the proof, we get:

**Corollary 3.6.** *If there are no consecutive* 1*'s in* $\mathbf{k}$ *we have* $\zeta_{\mathcal{S}}^*(\mathbf{k}) = \zeta_{\mathcal{S}}^{ш}(\mathbf{k})$.

*Proof of Proposition 1.29 (ii).* {♨ **Todo:** Sketch the proof and give reference♨}. □

**Theorem 3.7.** *For all $w, v \in \mathfrak{H}^1$ we have*

$$\zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(w \sqcup\!\sqcup v) = (-1)^{\mathrm{wt}(w)} \zeta_{\mathcal{S}}^{\sqcup\!\sqcup}(\overline{w} v)\,.$$

*Proof.* {♨ **Todo:** Give reference ♨} □

Until here in lecture 10 (8th July, 2022)

# §4 BTT-Philosophy and $\mathcal{Q}$-multiple zeta values

We will now come back to the multiple harmonic sums at roots of unity. For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ and $m \geq 1$ these were defined by

$$H_m(\mathbf{k}; q) = H_m(k_1, \dots, k_r; q) = \sum_{m \geq m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1} \cdots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}} \in \mathbb{Q}[[q]]\,. \tag{4.1}$$

As explained in Section 1.4, we are interested in the values $H_{n-1}(\mathbf{k}; \zeta_n) \in \mathbb{Q}(\zeta_n)$, since these can be related to symmetric and finite multiple zeta values. The connection to finite multiple zeta values was given by Theorem 1.37, which stated that for any primitive $p$-th root of unity $\zeta_p$, we have

$$(H_{p-1}(\mathbf{k}; \zeta_p) \mod \mathfrak{p})_p = \zeta_{\mathcal{A}}(\mathbf{k})\,,$$

where $\mathfrak{p} = (1 - \zeta_p)$ is the prime ideal of $\mathbb{Z}[\zeta_p]$ generated by $1 - \zeta_p$. In the next section we will proof the connection to symmetric multiple zeta values and then define $\mathcal{Q}$-multiple zeta values.

## 4.1 The connection to symmetric multiple zeta values

In this section, we want to sketch the proof of Theorem 1.36, i.e. we want to show that for any index set $\mathbf{k} = (k_1, \dots, k_r)$ the limit $\xi(\mathbf{k}) := \lim_{n\to\infty} H_{n-1}(\mathbf{k}; e^{\frac{2\pi i}{n}})$ exists and we have

$$\xi(\mathbf{k}) = \lim_{n\to\infty} H_{n-1}(\mathbf{k}; e^{\frac{2\pi i}{n}}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta^*\big(k_a, k_{a-1}, \dots, k_1; \frac{\pi i}{2}\big) \zeta^*(k_{a+1}, k_{a+2}, \dots, k_r; -\frac{\pi i}{2})\,.$$

In particular, this shows

$$\mathrm{Re}\,(\xi(\mathbf{k})) \equiv \zeta_{\mathcal{S}}(\mathbf{k}) \mod \pi^2 \mathcal{Z}\,.$$

The proof of Theorem 1.36 was first given in [BTT], which we will sketch in the following.

**Lemma 4.1.** *For any index $\mathbf{k} = (k_1, \dots, k_r)$ the polynomial*

$$R_{\mathbf{k}}(X; T) = \sum_{j=0}^{r} (-1)^{k_1+\cdots+k_j} \zeta^*(k_j, k_{j-1}, \dots, k_1; T + X) \zeta^*(k_{j+1}, \dots, k_r; T - X)$$

*does not depend on $T$, i.e. $R_{\mathbf{k}}(X; T) = R_{\mathbf{k}}(X; 0)$.*

*Proof.* This is a generalization of the $*$-part of Proposition 1.29 (i) and can be proven in a similar way. Exercise 15. □

We rewrite the value $H_{n-1}(\mathbf{k}; e^{2\pi i/n})$. Let $n$ be a positive integer. When $q = e^{2\pi i/n}$ we see that

$$\frac{1}{[m]_q} = \frac{1-q}{1-q^m} = e^{-\frac{\pi i}{n}(m-1)} \frac{\sin\frac{\pi}{n}}{\sin\frac{m\pi}{n}} \qquad (n > m \geq 0).$$

Therefore it holds that

$$H_{n-1}(\mathbf{k}; e^{\frac{2\pi i}{n}}) = \left(e^{\frac{\pi i}{n}} \frac{n}{\pi} \sin\frac{\pi}{n}\right)^{\mathrm{wt}(\mathbf{k})} \sum_{n>m_1>\cdots>m_r>0} \prod_{j=1}^{r} \frac{e^{\frac{\pi i}{n}(k_j-2)m_j}}{\left(\frac{n}{\pi}\sin\frac{m_j\pi}{n}\right)^{k_j}}$$

for any non-empty index $\mathbf{k} = (k_1, \ldots, k_r)$. Decompose the set $\{(m_1, \ldots, m_r) \in \mathbb{Z}^r \mid n > m_1 > \cdots > m_r > 0\}$ into the disjoint union

$$\bigsqcup_{a=0}^{r} \{(m_1, \ldots, m_r) \in \mathbb{Z}^r \mid n > m_1 > \cdots > m_a > \frac{n}{2} \geq m_{a+1} > \cdots > m_r > 0\}$$

and change the summation variables $m_j$ to $n_j = n - m_{a+1-j}$ $(1 \leq j \leq a)$ and $l_j = m_{a+j}$ $(1 \leq j \leq r-a)$. Then we find that

$$H_{n-1}(\mathbf{k}; e^{\frac{2\pi i}{n}}) = \left(e^{\frac{\pi i}{n}} \frac{n}{\pi} \sin\frac{\pi}{n}\right)^{\mathrm{wt}(\mathbf{k})}$$
$$\times \sum_{a=0}^{r} (-1)^{\sum_{j=1}^{a} k_j} \sum_{n/2>n_1>\cdots>n_a>0} \prod_{j=1}^{a} \frac{e^{-\frac{\pi i}{n}(k_{a+1-j}-2)n_j}}{\left(\frac{n}{\pi}\sin\frac{n_j\pi}{n}\right)^{k_{a+1-j}}} \sum_{n/2\geq l_1>\cdots>l_{r-a}>0} \prod_{j=1}^{r-a} \frac{e^{\frac{\pi i}{n}(k_{a+j}-2)l_j}}{\left(\frac{n}{\pi}\sin\frac{l_j\pi}{n}\right)^{k_{a+j}}}.$$

Motivated by the above expression we introduce the following numbers. For an index $\mathbf{k} = (k_1, \ldots, k_r)$ and a positive integer $n$, we define

$$A_n^-(\mathbf{k}) = \sum_{n/2>m_1>\cdots>m_r>0} \prod_{j=1}^{r} \frac{e^{-\frac{\pi i}{n}(k_j-2)m_j}}{\left(\frac{n}{\pi}\sin\frac{m_j\pi}{n}\right)^{k_j}},$$
$$A_n^+(\mathbf{k}) = \sum_{n/2\geq m_1>\cdots>m_r>0} \prod_{j=1}^{r} \frac{e^{\frac{\pi i}{n}(k_j-2)m_j}}{\left(\frac{n}{\pi}\sin\frac{m_j\pi}{n}\right)^{k_j}}.$$

Then we see that

$$H_{n-1}(\mathbf{k}; e^{\frac{2\pi i}{n}}) = \left(e^{\frac{\pi i}{n}} \frac{n}{\pi} \sin\frac{\pi}{n}\right)^{\mathrm{wt}(\mathbf{k})} \sum_{a=0}^{r} (-1)^{\sum_{j=1}^{a} k_j} A_n^-(k_a, k_{a-1}, \ldots, k_1) A_n^+(k_{a+1}, k_{a+2}, \ldots, k_r).$$

Notice that we have

$$A_n^-(k_1, \ldots, k_r) = \begin{cases} \overline{A_n^+(k_1, \ldots, k_r)} & (n\text{: odd}), \\ \overline{A_n^+(k_1, \ldots, k_r)} + (-\frac{\pi i}{n})^{k_1} \overline{A_n^+(k_2, \ldots, k_r)} & (n\text{: even}), \end{cases}$$

where the bar on the right-hand side denotes complex conjugation. We want to understand the behavior of $A_n^+$ for $n \to \infty$. For this we will first consider the cases where the index $\mathbf{k}$ is admissible.

**Lemma 4.2** ([BTT] Lemma 2.7). *Let $\mathbf{k}$ be an admissible index. Then it holds that*

$$A_n^+(\mathbf{k}) = \zeta(\mathbf{k}) + O\left(\frac{(\log n)^{J_1(\mathbf{k})}}{n}\right) \qquad (n \to +\infty),$$

*where $J_1(\mathbf{k})$ is a positive integer which depends on $\mathbf{k}$.*

*Proof.* Set $\mathbf{k} = (k_1, \dots, k_r)$ and define for $k \geq 1$ the function

$$g_k(x) = e^{(k-2)ix} \left(\frac{x}{\sin x}\right)^k .$$

Then it holds that $|A_n^+(\mathbf{k}) - \zeta(\mathbf{k})| \leq I_1 + I_2$, where

$$I_1 = \sum_{n/2 \geq m_1 > \cdots > m_r > 0} \prod_{j=1}^r \frac{1}{m_j^{k_j}} \left| \prod_{j=1}^r g_{k_j}\left(\frac{m_j \pi}{n}\right) - 1 \right|,$$
$$I_2 = \sum_{m > n/2} \frac{1}{m^{k_1}} \left( \sum_{m > m_2 > \cdots > m_r > 0} \prod_{j=2}^r \frac{1}{m_j^{k_j}} \right).$$

Since $g_k(x) = 1 + (k-2)ix + o(x)$ $(x \to +0)$, there exists a positive constant $C$ depending on $k$ such that $|g_k(m\pi/n) - 1| \leq Cm/n$ for all integers $m$ and $n$ satisfying $n/2 \geq m > 0$. Using the identity

$$\left( \prod_{j=1}^r x_j \right) - 1 = \sum_{a=1}^r \left(\prod_{j=1}^{a-1} x_j\right)(x_a - 1)$$

and the inequality $0 < (\sin x)^{-1} \leq \pi/2x$ on the interval $(0, \frac{\pi}{2}]$, we see that

$$\begin{aligned} I_1 &\leq \frac{C_1}{n} \sum_{a=1}^r \sum_{n/2 \geq m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_a^{k_a - 1} \cdots m_r^{k_r}} \\ &\leq \frac{C_1}{n} \sum_{a=1}^r \sum_{n/2 \geq m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1 - 1} m_2^{k_2} \cdots m_r^{k_r}} \\ &= \frac{C_1 r}{n} \sum_{n/2 \geq m > 0} \frac{1}{m^{k_1 - 1}} \left( \sum_{m > m_2 > \cdots > m_r > 0} \prod_{j=2}^r \frac{1}{m_j^{k_j}} \right) \end{aligned}$$

for some positive constant $C_1$ which depends on $\mathbf{k}$. Using the estimation

$$\sum_{m > m_2 > \cdots > m_r > 0} \prod_{j=2}^r \frac{1}{m_j^{k_j}} \leq \left( \sum_{s=1}^{m-1} \frac{1}{s} \right)^{r-1} \leq (2 \log m)^{r-1},$$

we get

$$I_1 + I_2 \leq C_2 \left( \frac{1}{n} \sum_{n/2 > m > 0} \frac{(\log m)^{r-1}}{m^{k_1 - 1}} + \sum_{m > n/2} \frac{(\log m)^{r-1}}{m^{k_1}} \right)$$

for some positive constant $C_2$ which depends on $\mathbf{k}$. Since $k_1 \geq 2$ it holds that

$$\sum_{n/2 > m > 0} \frac{(\log m)^{r-1}}{m^{k_1 - 1}} = O((\log n)^r), \qquad \sum_{m > n/2} \frac{(\log m)^{r-1}}{m^{k_1}} = O\left( \frac{(\log n)^{r-1}}{n} \right)$$

as $n \to +\infty$. This completes the proof. □

**Lemma 4.3** ([BTT] Lemma 2.8)**.** *We have*

$$A_n^+(1) = \log\left(\frac{n}{\pi}\right) + \gamma - \frac{\pi i}{2} + O\left(\frac{1}{n}\right) \qquad (n \to +\infty).$$

*Proof.* From the definition of $A_n^+(1)$ we see that

$$A_n^+(1) = \frac{\pi}{n} \sum_{n/2 \geq m > 0} \left( \frac{\cos \frac{m\pi}{n}}{\sin \frac{m\pi}{n}} - i \right) = \frac{\pi}{n} \sum_{n/2 \geq m > 0} \frac{\cos \frac{m\pi}{n}}{\sin \frac{m\pi}{n}} - \frac{\pi i}{2} + O\left(\frac{1}{n}\right)$$

as $n \to +\infty$. Hence it suffices to show that

$$\frac{\pi}{n} \sum_{n/2 \geq m > 0} \frac{\cos \frac{m\pi}{n}}{\sin \frac{m\pi}{n}} = \log \frac{n}{\pi} + \gamma + O\left(\frac{1}{n}\right) \qquad (n \to +\infty). \tag{4.2}$$

Since the function $f(x) = x^{-1} - (\tan x)^{-1}$ is positive and increasing on the interval $(0, \pi)$, we see that

$$\int_0^{\frac{n-1}{2}} f\Big(\frac{\pi x}{n}\Big)\, dx \leq \sum_{n/2 \geq m > 0} \left( \frac{n}{\pi} \frac{1}{m} - \frac{\cos \frac{m\pi}{n}}{\sin \frac{m\pi}{n}} \right) \leq \int_1^{\frac{n}{2}+1} f\Big(\frac{\pi x}{n}\Big)\, dx.$$

Set $g(x) = \log(1+x) - \log(\cos \frac{\pi x}{2})$. By direct calculation we have

$$\int_0^{\frac{n-1}{2}} f\Big(\frac{\pi x}{n}\Big)\, dx = \frac{n}{\pi} \left( g\Big(-\frac{1}{n}\Big) + \log \frac{\pi}{2} \right),$$
$$\int_1^{\frac{n}{2}+1} f\Big(\frac{\pi x}{n}\Big)\, dx = \frac{n}{\pi} \left( g\Big(\frac{2}{n}\Big) + \log\left(\frac{n}{\pi} \sin \frac{\pi}{n}\right) + \log \frac{\pi}{2} \right).$$

Since $g(x) = x + o(x)$ $(x \to 0)$ and $\log(x^{-1} \sin x) = o(x)$ $(x \to +0)$, there exist positive constants $c_1$ and $c_2$ such that

$$\int_0^{\frac{n-1}{2}} f\Big(\frac{\pi x}{n}\Big)\, dx \geq -c_1 + \frac{n}{\pi} \log \frac{\pi}{2}, \quad \int_1^{\frac{n}{2}+1} f\Big(\frac{\pi x}{n}\Big)\, dx \leq c_2 + \frac{n}{\pi} \log \frac{\pi}{2}$$

for $n \gg 0$. Therefore we find that

$$\frac{\pi}{n} \sum_{n/2 \geq m > 0} \frac{\cos \frac{m\pi}{n}}{\sin \frac{m\pi}{n}} = \sum_{n/2 \geq m > 0} \frac{1}{m} - \log \frac{\pi}{2} + O\Big(\frac{1}{n}\Big) \qquad (n \to +\infty).$$

Using the asymptotic expansion

$$\sum_{n/2 \geq m > 0} \frac{1}{m} = \log \frac{n}{2} + \gamma + O\Big(\frac{1}{n}\Big) \qquad (n \to +\infty),$$

we get the formula (4.2). □

**Proposition 4.4.** *For any index* $\mathbf{k}$ *it holds that*

$$A_n^{\pm}(\mathbf{k}) = \zeta^*\left(\mathbf{k}; \log\left(\frac{n}{\pi}\right) + \gamma \mp \frac{\pi i}{2}\right) + O\left(\frac{(\log n)^{J(\mathbf{k})}}{n}\right) \qquad (n \to +\infty), \tag{4.3}$$

*where* $\gamma$ *is Euler's constant and* $J(\mathbf{k})$ *is a positive integer which depends on* $\mathbf{k}$.

*Proof.* Lemma 4.2 implies that the equality (4.3) for $A_n^+(\mathbf{k})$ holds if $\mathbf{k}$ is admissible. Let us prove that it holds also for the index $(\{1\}^s, \mathbf{k})$ with any $s \geq 0$ and any admissible index $\mathbf{k}$.
Using the equality

$$\frac{e^{-\frac{\pi i}{n}m}}{\frac{n}{\pi}\sin\frac{m\pi}{n}} \frac{e^{\frac{\pi i}{n}(k-2)m}}{\left(\frac{n}{\pi}\sin\frac{m\pi}{n}\right)^k} = \frac{e^{\frac{\pi i}{n}(k-1)m}}{\left(\frac{n}{\pi}\sin\frac{m\pi}{n}\right)^{k+1}} - \frac{2\pi i}{n}\frac{e^{\frac{\pi i}{n}(k-2)m}}{\left(\frac{n}{\pi}\sin\frac{m\pi}{n}\right)^k},$$

for $k \geq 1$ and $n/2 \geq m > 0$, we see that

$$\begin{aligned} A_n^+(1)A_n^+(\{1\}^s, \mathbf{k}) = (s+1)&A_n^+(\{1\}^{s+1}, \mathbf{k}) \\ &+ \sum_{b=1}^{s}\left(A_n^+(\{1\}^{b-1}, 2, \{1\}^{s-b}, \mathbf{k}) - \frac{2\pi i}{n}A_n^+(\{1\}^s, \mathbf{k})\right) \\ &+ \sum_{a=1}^{r}\left(A_n^+(\{1\}^s, \mathbf{k}'(a)) + A_n^+(\{1\}^s, \mathbf{k}''(a)) - \frac{2\pi i}{n}A_n^+(\{1\}^s, \mathbf{k})\right), \end{aligned}$$

where $\mathbf{k}'(a)$ and $\mathbf{k}''(a)$ are the indices defined by

$$\mathbf{k}'(a) = (k_1, \ldots, k_a + 1, \ldots, k_r), \quad \mathbf{k}''(a) = (k_1, \ldots, k_a, 1, k_{a+1}, \ldots, k_r)\,.$$

We obtain the desired equality (4.3) by induction on $s$ by the stuffle product and Lemma 4.3. □

*Proof of Theorem 1.36.* The statement follows by combining Lemma 4.1 and Proposition 4.4. □

## 4.2 Duality

In this section we want to give the proof of Theorem 1.39, i.e. we want to show that for all $n \geq 1$ and all indices $\mathbf{k}$ we have for any primitive $n$-th root of unity $\zeta_n$

$$H_{n-1}^\star(\mathbf{k}; \zeta_n) = (-1)^{\mathrm{wt}(\mathbf{k})+1} H_{n-1}^\star(\overline{\mathbf{k}^\vee}; \zeta_n)\,.$$

Here we define for an index $\mathbf{k} = (k_1, \ldots, k_r)$ its reverse by $\overline{\mathbf{k}} = (k_r, \ldots, k_1)$ and $\mathbf{k}^\vee$ is the Hoffman dual, which we defined in the end of Section 1.2.
We will use the following fact.

**Lemma 4.5.** *Suppose that $n \geq 1$ and $\zeta_n$ is a primitive $n$-th root of unity. Then it holds that $(-1)^n \zeta_n^{n(n+1)/2} = -1$.*

*Proof of Theorem 1.39.* Note that any index is uniquely written in the form

$$(\{1\}^{a_1-1}, b_1+1, \ldots, \{1\}^{a_{r-1}-1}, b_{r-1}+1, \{1\}^{a_r-1}, b_r), \tag{4.4}$$

where $r$ and $a_i, b_i$ $(1 \leq i \leq r)$ are positive integers [6]. Denote it by $[a_1, \ldots, a_r; b_1, \ldots, b_r]$. Then we see that

$$\overline{[a_1, \ldots, a_r; b_1, \ldots, b_r]^\vee} = [b_r, \ldots, b_1; a_r, \ldots, a_1].$$

Now we fix a positive integer $r$ and introduce the generating function

$$K(x_1, \ldots, x_r; y_1, \ldots, y_r) = \sum \frac{H_{n-1}^\star([a_1, \ldots, a_r; b_1, \ldots, b_r]; \zeta_n)}{(1-\zeta_n)^{a_1+\cdots+a_r+b_1+\cdots+b_r-1}} \prod_{i=1}^{r}(x_i^{a_i-1} y_i^{b_i-1}),$$

[6] If $r = 1$, (4.4) should read as $(\{1\}^{a_1-1}, b_1)$.

where the sum is taken over all positive integers $a_i, b_i$ $(1 \le i \le r)$. Then Theorem 1.39 follows from the equality

$$K(x_1,\dots,x_r;y_1,\dots,y_r) = K(-y_r,\dots,-y_1;-x_r,\dots,-x_1). \tag{4.5}$$

Let us prove (4.5). It holds that

$$1+\sum_{a=2}^{\infty}\sum_{B\ge m_1\ge\cdots\ge m_{a-1}\ge A}\frac{x^{a-1}}{\prod_{i=1}^{a-1}(1-\zeta_n^{m_i})} = \prod_{i=A}^{B}\frac{1-\zeta_n^i}{1-x-\zeta_n^i}$$

for $n > B \ge A > 0$, and that

$$\sum_{b=1}^{\infty}\frac{\zeta_n^{bm}}{(1-\zeta_n^m)^{b+1}}y^{b-1} = \frac{1}{1-\zeta_n^m}\frac{\zeta_n^m}{1-\zeta_n^m(1+y)}$$

for $n > m > 0$. Using the above formulas we have

$$\begin{aligned}K(x_1,\dots,x_r;y_1,\dots,y_r) &= \sum_{n>l_1\ge\cdots\ge l_r>0}\prod_{i=l_r}^{n-1}(1-\zeta_n^i)\\
&\times\prod_{j=1}^{r-1}\left(\frac{\zeta_n^{l_j}}{1-\zeta_n^{l_j}(1+y_j)}\prod_{i=l_j}^{l_{j-1}}\frac{1}{1-x_j-\zeta_n^i}\right)\frac{1}{1-\zeta_n^{l_r}(1+y_r)}\prod_{i=l_r}^{l_{r-1}}\frac{1}{1-x_r-\zeta_n^i},\end{aligned}$$

where $l_0 = n-1$. Rewrite the right-hand side above by using the partial fraction expansion

$$\begin{aligned}\prod_{i=A}^{B}\frac{1}{X-\zeta_n^i} &= \sum_{i=A}^{B}\frac{1}{X-\zeta_n^i}\prod_{j=A}^{i-1}\frac{1}{\zeta_n^i-\zeta_n^j}\prod_{j=i+1}^{B}\frac{1}{\zeta_n^i-\zeta_n^j}\\
&= \sum_{t=A}^{B}\frac{1}{X-\zeta_n^t}\frac{(-1)^{B-t}\zeta_n^{-\binom{B+1}{2}+At-\binom{t}{2}}}{\prod_{i=1}^{t-A}(1-\zeta_n^{-i})\prod_{i=1}^{B-t}(1-\zeta_n^{-i})}\end{aligned}$$

for $n > B \ge A > 0$. Then we find that

$$\begin{aligned}&K(x_1,\dots,x_r;y_1,\dots,y_r)\\
&= \sum_{n>t_1\ge l_1\ge\cdots\ge t_r\ge l_r>0}\prod_{i=l_r}^{n-1}(1-\zeta_n^i)(-1)^{\sum_{j=1}^r(l_{j-1}-t_j)}\zeta_n^{\sum_{j=1}^r\left(-\binom{l_{j-1}+1}{2}+l_jt_j-\binom{t_j}{2}\right)}\\
&\qquad\times\prod_{j=1}^{r}\left(\prod_{i=1}^{t_j-l_j}\frac{1}{1-\zeta_n^{-i}}\prod_{i=1}^{l_{j-1}-t_j}\frac{1}{1-\zeta_n^{-i}}\right)\\
&\qquad\times\prod_{j=1}^{r-1}\left(\frac{\zeta_n^{l_j}}{1-\zeta_n^{l_j}(1+y_j)}\frac{1}{1-x_j-\zeta_n^{t_j}}\right)\frac{1}{1-\zeta_n^{l_r}(1+y_r)}\frac{1}{1-x_r-\zeta_n^{t_r}}.\end{aligned}$$

Now change the summation variable $t_j$ and $l_j$ to $n-l_{r+1-j}$ and $n-t_{r+1-j}$, respectively $(1 \le j \le r)$. As a result we get the desired equality (4.5) using Lemma 4.5. □

Until here in lecture 11 (15th July, 2022)

### 4.3 Reversal relation & Linear shuffle relations

We will now describe the linear shuffle relations for the multiple harmonic sums at roots of unity. Recall that we set $\overline{\mathbb{N}} = \{\bar{1}\} \cup \mathbb{Z}_{\geq 1}$ and then define for $k \in \overline{\mathbb{N}}$ and $m \geq 1$ the following $q$-series

$$f_k(m) = \begin{cases} \frac{q^m}{[m]_q}, & \text{if } k = \bar{1} \\ \frac{q^{(k-1)m}}{[m]_q^k}, & \text{if } k \in \mathbb{Z}_{\geq 1} \end{cases} . \tag{4.6}$$

With this we can generalize the definition of the multiple harmonic $q$-series for $\mathbf{k} = (k_1, \dots, k_r) \in \overline{\mathbb{N}}^r$ and $m \geq 1$ by

$$H_m(\mathbf{k}; q) = H_m(k_1, \dots, k_r; q) = \sum_{m \geq m_1 > \dots > m_r > 0} \prod_{j=1}^{r} f_{k_j}(m_j) \in \mathbb{Q}[[q]] .$$

These can be viewed for any $m \geq 1$ as $\mathcal{C}$-algebra homomorphims

$$\begin{aligned} H_m(\cdot; q) : \widehat{\mathfrak{H}}^1_{*_q} &\longrightarrow \mathbb{Q}[[q]], \\ e_{k_1} \dots e_{k_r} &\longmapsto H_m(k_1, \dots, k_r; q) , \end{aligned}$$

where we defined $\widehat{\mathfrak{H}}^1 = \mathcal{C}\langle e_{\bar{1}}, e_1, e_2, e_3, \dots \rangle$ and $\widehat{\mathfrak{H}}^1_{*_q}$ is $\widehat{\mathfrak{H}}^1$ equipped with $q$-stuffle product. Now recall that for $k \geq 2$ we defined

$$e_{\overline{k}} := \sum_{j=2}^{k} \binom{k-2}{j-2} \hbar^{k-j} e_j . \tag{4.7}$$

which was inspired by the equation

$$\frac{q^m}{[m]_q^k} = \sum_{j=2}^{k} \binom{k-2}{j-2} (1-q)^{k-j} \frac{q^{(j-1)m}}{[m]_q^j} , \qquad (k \geq 2). \tag{4.8}$$

The next lemma shows, that the $e_{\overline{k}}$ will be used when considering the reverse of a word (doing the change of summation $m \to n-m$), when evaluation the $H_{n-1}$ at primitive $n$-th roots of unity.

**Lemma 4.6.** *Let $q = \zeta_n$ be primitive $n$-th root of unity and $n > m$. Then we have for $k \geq 1$*

$$\frac{q^{(k-1)(n-m)}}{[n-m]_q^k} = (-1)^k \frac{q^m}{[m]_q^k} .$$

*Proof.* By direct calculation we get $[n-m]_q = -q^{-m}[m]_q$ from which the statement follows. □

Lemma 4.6 can be seen as an analogue of the equation $\frac{1}{m^k} \equiv \frac{(-1)^k}{(p-m)^k} \mod p$, which we used in the proof of the reversal relation and the linear shuffle relations of finite multiple zeta values.

**Definition 4.7.** *We define the anti-automorphism*[7] $\psi : \widehat{\mathfrak{H}}^1 \to \widehat{\mathfrak{H}}^1$ *for $k \geq 1$ by*

$$\psi(e_k) = e_{\overline{k}}, \quad \psi(e_{\bar{1}}) = e_1 ,$$

*i.e. for $k_1, \dots, k_r \in \overline{\mathbb{N}}$ we have $\psi(e_{k_1} \dots e_{k_r}) = e_{\overline{k_r}} \cdots e_{\overline{k_1}}$ with the convention $e_{\overline{\bar{1}}} = e_1$.*

[7] with respect to the concatenation product

One can check that $\psi$ is an involution and as a consequence of Lemma4.6 we get the following analogue of the reversal formula:

**Proposition 4.8.** *For any primitive $n$-th root of unity $\zeta_n$ and $w \in \widehat{\mathfrak{H}}^1$ we have*

$$H_{n-1}(w;\zeta_n) = (-1)^{\mathrm{wt}(w)} H_{n-1}(\psi(w);\zeta_n)\,.$$

*Proof.* {☕ **Todo:** ...☕} □

As for finite multiple zeta values, this reversal relation is just a special case of the linear shuffle relations.

**Theorem 4.9.** *For $n \geq 2$ and any primitive $n$-th root of unity $\zeta_n$ and $w, v \in \widehat{\mathfrak{H}}^1$ we have*

$$H_{n-1}(w \sqcup\!\sqcup_q v;\zeta_n) = (-1)^{\mathrm{wt}(w)} H_{n-1}(\psi(w)v;\zeta_n)\,.$$

*Proof.* This can be shown in a similar way as the linear shuffle relations for finite multiple zeta values (Theorem 3.1) together with Proposition 2.31 (Exercise 16). □

## 4.4 $\mathcal{Q}$-multiple zeta values

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# q-analogues and finite multiple zeta values

## Exercises

Topics in Mathematical Science III (Nagoya University, Spring 2022)

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**Exercise 1.** Use the finite double shuffle relations to show $\zeta(6) = 6\zeta(3,3) - 3\zeta(4,2)$.

**Exercise 2.** Find a formula for the number of indices and admissible indices of a given weight (and depth), i.e. for $r, k \geq 0$ find explicit expressions for the following four integers

$$I_k = \sum_{r\geq 0} I_{k,r}\,, \qquad I_{k,r} = \left|\{\mathbf{k} \in \mathbb{Z}_{\geq 1}^r \mid \mathrm{wt}(\mathbf{k}) = k\}\right|,$$
$$I_k^0 = \sum_{r\geq 0} I_{k,r}^0\,, \qquad I_{k,r}^0 = \left|\{\mathbf{k} \in \mathbb{Z}_{\geq 1}^r \mid \mathrm{wt}(\mathbf{k}) = k, \mathbf{k} \text{ is admissible}\}\right|.$$

**Exercise 3.**

(i) Show that Conjecture 1.13 together with Proposition 1.12 would imply that all multiple zeta values (except for $\zeta(\emptyset) = 1$) are transcendental.

(ii) Show that Conjecture 1.17 (Hoffman) would imply Conjecture 1.15 (Zagier).

**Exercise 4.** Show that for any $k_1, \dots, k_r \in \mathbb{Z}$ we have $\zeta_{\mathcal{A}}(k_1, \dots, k_r) \in \mathcal{Z}^{\mathcal{A}}$, i.e. show that you can write $\zeta_{\mathcal{A}}(k_1, \dots, k_r)$ as a linear combination of finite multiple zeta values with positive entries.

**Exercise 5.** Show that for all $m \geq 1$ and $k_1, \dots, k_r \geq 1$ we have

$$S_m(k_1, \dots, k_r) = \sum_{j=0}^{r} (-1)^{k_1+\dots+k_j} H_m(k_j, k_{j-1}, \dots, k_1) H_m(k_{j+1}, \dots, k_r)\,,$$

where $S_m$ is defined by (1.12) and $H_m$ by (4.1).

**Exercise 6.** Let $\zeta_p$ be primitive $p$-th root of unity and consider the ideal $\mathfrak{p} = (1 - \zeta_p)$ of $\mathbb{Z}[\zeta_p]$.

(i) Show that $\mathfrak{p}$ is prime.

(ii) Show that $\mathbb{Z}[\zeta_p]/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$.

**Exercise 7.**

(i) Prove Proposition 1.38, i.e. show that for all $n, k \geq 1$ we have

$$H_{n-1}(k; e^{\frac{2\pi i}{n}}) = -\frac{b_k(n)}{k!}\left(n(1 - e^{\frac{2\pi i}{n}})\right)^k.$$

(ii) Show that $b_k(n)\, n^k \in \mathbb{Q}[n]$.

(iii) Use (i) to give another proof of $\zeta_{\mathcal{A}}(k)=0$ and $\zeta_{\mathcal{S}}(k)=0$ by using Theorem 1.36 and 1.37.

**Exercise 8.** Show that the quasi-shuffle product $*_\diamond$ defined in Definition 2.1 is associative.

**Exercise 9.**

(i) Show that for any $k\geq 1$ and $m\geq 1$ we have

$$1+\sum_{n=1}^{\infty} H_m(\overbrace{k,\dots,k}^{n})X^n = \exp\left(\sum_{n=1}^{\infty}(-1)^{n-1}H_m(nk)\frac{X^n}{n}\right).$$

(ii) Prove that for $k\geq 1$ we have $\zeta_{\mathcal{A}}(k,\dots,k)=0$ and $\zeta(2k,\dots,2k)\in\mathbb{Q}[\pi^2]$.

**Exercise 10.** Show that $\mathfrak{H}^1_* = \mathfrak{H}^0_*[z_1]$.

**Exercise 11.** Prove Proposition 2.26, i.e. show that for any $w\in\widehat{\mathfrak{H}}^1$ we have $\mathrm{Li}^q_w = w1$.

**Exercise 12.** Show that for $f,g\in z\mathbb{Q}[[q,z]]$ we have

$$\begin{aligned}(af)(ag) &= a((af)g + f(ag) + \hbar fg),\\ (bf)g &= f(bg) = b(fg)\,.\end{aligned}$$

**Exercise 13.**

(i) Determine $\zeta_q(e_{\bar{1}} ⧢_q e_{\hat{2}} - e_{\bar{1}} *_q e_{\hat{2}})$ and consider the limit $q\to 1$ if possible.

(ii) Calculate $\sigma(\sigma(e_{\hat{2}}) *_q \sigma(e_{\hat{2}}))$ and compare it with $e_{\hat{2}} ⧢_q e_{\hat{2}}$.

**Exercise 14.** For $k\geq 1$ show that $\zeta_{\mathcal{A}}(k)=0$ by just using the equations $(*)$ and $(⧢)$ in Conjecture 3.3.

**Exercise 15.** Prove Lemma 4.1, i.e. show that for any index $\mathbf{k}=(k_1,\dots,k_r)$ the polynomial

$$R_{\mathbf{k}}(X;T) = \sum_{j=0}^{r}(-1)^{k_1+\dots+k_j}\zeta^*(k_j,k_{j-1},\dots,k_1;T+X)\zeta^*(k_{j+1},\dots,k_r;T-X)$$

does not depend on $T$.

**Exercise 16.** Prove the linear shuffle relations for multiple harmonic $q$-series at roots of unity (Theorem 4.9), i.e. show that for any primitive $n$-th root of unity $\zeta_n$ and $w,v\in\widehat{\mathfrak{H}}^1$ we have

$$H_{n-1}(w ⧢_q v;\zeta_n) = (-1)^{\mathrm{wt}(w)}H_{n-1}(\psi(w)v;\zeta_n)\,.$$

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