Finite multiple harmonic q-series at roots of unity and finite & symmetric multiple zeta values

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For an index $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$ with $k_1 \geq 2$ the multiple zeta value and the multiple zeta star value are defined by

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}, \qquad \zeta^\star(\mathbf{k}) = \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

We denote by $\mathcal{Z}$ the $\mathbb{Q}$-vector space spanned by all multiple zeta values. The space $\mathcal{Z}$ forms a subalgebra of $\mathbb{R}$ over $\mathbb{Q}$ and is the same space with that spanned by multiple zeta star values. In their recent work [3], Kaneko and Zagier introduce two variants of these values, called finite and symmetric multiple zeta values, and give a mysterious conjecture relating them.

Finite multiple zeta values are defined for an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$ by

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \left(\sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_p \in \mathcal{A},$$

where $\mathcal{A}$ is the $\mathbb{Q}$-algebra $\mathcal{A} = (\prod_p \mathbb{F}_p) / (\bigoplus_p \mathbb{F}_p)$ with p running over all primes.

Symmetric multiple zeta values are defined for an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{\geq 1}^r$ by

$$\zeta_{\mathcal{S}}(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1 + \cdots + k_a} \zeta^*(k_a, , k_{a-1}, \dots, k_1) \zeta^*(k_{a+1}, k_{a+2}, \dots, k_r) \in \mathcal{Z}/\zeta(2)\mathcal{Z},$$

where ζ^* denotes the regularized multiple zeta values defined in [2]. Both, finite and symmetric multiple zeta values, also have star-versions $\zeta_{\mathcal{A}}^\star(\mathbf{k}) \in \mathcal{A}$ and $\zeta_{\mathcal{S}}^\star(\mathbf{k}) \in \mathcal{Z}/\zeta(2)\mathcal{Z}$, which relate to their non-star version like $\zeta^\star$ does to ζ. A priori finite and symmetric multiple zeta values are completely different objects, but Kaneko and Zagier gave the following surprising conjecture.

Conjecture 1 *(Kaneko–Zagier [3]) Let $\mathcal{Z}_{\mathcal{A}}$ be the $\mathbb{Q}$-vector space of finite multiple zeta values. There exists a $\mathbb{Q}$-algebra isomorphism*

$$\varphi_{KZ} : \mathcal{Z}_{\mathcal{A}} \longrightarrow \mathcal{Z}/\zeta(2)\mathcal{Z},$$
$$\zeta_{\mathcal{A}}(\mathbf{k}) \longmapsto \zeta_{\mathcal{S}}(\mathbf{k}) \mod \zeta(2)\mathcal{Z}.$$

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In our work [1], we study this mysterious relationship between finite and symmetric multiple zeta values by considering the *finite multiple harmonic q-series*, defined for $n \in \mathbb{Z}_{\geq 1}$ and an index $\mathbf{k} = (k_1, \dots, k_r)$ by

$$z_n(\mathbf{k}; q) = \sum_{n > m_1 > \cdots > m_r > 0} \frac{q^{(k_1-1)m_1} \dots q^{(k_r-1)m_r}}{[m_1]_q^{k_1} \dots [m_r]_q^{k_r}},$$

where $[m]_q = \frac{1-q^m}{1-q}$ denotes the usual q-integer. Again, its star version $z_n^\star(\mathbf{k}; q)$ is defined by allowing equalities among the m_j. In [1] it was observed that these sums relate to finite and symmetric multiple zeta values when evaluated at a primitive n-th root of unity $q = \zeta_n$, which gives elements in the cyclotomic field $\mathbb{Q}(\zeta_n)$. The connection to finite and symmetric multiple zeta values are then given by the following two theorems.

Theorem 1 *[1, Theorem 1.1] For any index $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^r$, we have*

$$(z_p(\mathbf{k}; \zeta_p) \mod \mathfrak{p}_p)_p = \zeta_{\mathcal{A}}(\mathbf{k}), \qquad (z_p^\star(\mathbf{k}; \zeta_p) \mod \mathfrak{p}_p)_p = \zeta_{\mathcal{A}}^\star(\mathbf{k}),$$

where $\mathfrak{p}_p = (1 - \zeta_p)$ is the prime ideal of $\mathbb{Z}[\zeta_p]$ generated by $1 - \zeta_p$.

Theorem 2 *[1, Theorem 1.2] For any index $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^r$, the limits*

$$\xi(\mathbf{k}) = \lim_{n \to \infty} z_n(\mathbf{k}; e^{2\pi i/n}), \qquad \xi^\star(\mathbf{k}) = \lim_{n \to \infty} z_n^\star(\mathbf{k}; e^{2\pi i/n})$$

exist in $\mathbb{C}$ and it holds that

$$\operatorname{Re} \xi(\mathbf{k}) \equiv \zeta_{\mathcal{S}}(\mathbf{k}), \qquad \operatorname{Re} \xi^\star(\mathbf{k}) \equiv \zeta_{\mathcal{S}}^\star(\mathbf{k})$$

modulo $\zeta(2)\mathcal{Z}$.

Relations among the z_n (resp. $z_n^\star$) therefore give relations among $\zeta_{\mathcal{A}}$ (resp. $\zeta_{\mathcal{A}}^\star$) and $\zeta_{\mathcal{S}}$ (resp. $\zeta_{\mathcal{S}}^\star$) of the same shape. This gives evidence towards the Conjecture of Kaneko and Zagier above. For example, one can show that we have for all $n \in \mathbb{Z}_{\geq 1}$

$$2z_n^\star(4,1;\zeta_n) + z_n^\star(3,2;\zeta_n) = \frac{(n^4-1)(n+5)}{1440}(1-\zeta_n)^5 + \frac{n+2}{3}(1-\zeta_n)^2 z_n^\star(2,1;\zeta_n),$$

which by the above Theorems implies the relations

$$2\zeta_{\mathcal{A}}^\star(4,1) + \zeta_{\mathcal{A}}^\star(3,2) = 0 \quad \text{and} \quad 2\zeta_{\mathcal{S}}^\star(4,1) + \zeta_{\mathcal{S}}^\star(3,2) \equiv 0 \mod \zeta(2)\mathcal{Z}.$$

It is therefore of great interest to study the relations among the finite multiple harmonic q-series at primitive roots of unity. One such family is given by the following theorem.

Theorem 3 *[1, Theorem 1.3] For any index $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^r$ and any n-th primitive root of unity ζ_n, we have*

$$z_n^\star(\mathbf{k}; \zeta_n) = (-1)^{\mathrm{wt}(\mathbf{k})+1} z_n^\star(\overline{\mathbf{k}^\vee}; \zeta_n),$$

where $\overline{\mathbf{k}^\vee}$ is the reverse of the Hoffman dual $\mathbf{k}^\vee$ (see [1, Section 2.4]).

This Theorem together with the results above gives a new proof of the duality formulas for $\zeta_{\mathcal{A}}^\star(\mathbf{k})$ and $\zeta_{\mathcal{S}}^\star(\mathbf{k})$, previously obtained by Hoffman and Jarossay respectively.

References

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