Formal multiple Eisenstein series (FMES) and their derivations

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The goal of this talk is to understand the following:

① Quasimodular forms (QMF) $\circlearrowleft sl_2$, $\ni G_k$

③ Multiple Eisenstein series (MES) $\mathbb{G}_{k_1,\dots,k_r}$

④ FMES $G(k_1,\dots,k_r)$ $\circlearrowleft sl_2$

$$\begin{tikzcd}
S^f \arrow[r, hook] \arrow[d] & M^f \arrow[r, hook] & \widetilde{M}^f \arrow[r, hook] \arrow[d, "\text{const. term}"'] & \mathcal{E}^f \arrow[r, hook] \arrow[d] & \mathfrak{g}_0^f \arrow[r, hook] & \mathfrak{g}^f \arrow[dll, "\pi"'] \\
0 \arrow[rr] & & \mathbb{Q}[\zeta^f(2)] \arrow[r, hook] & \mathcal{Z}^f & &
\end{tikzcd}$$

$\mathfrak{g}^f \ni G\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$, $\mathfrak{g}^f \circlearrowleft sl_2$

$\zeta(k_1,\dots,k_r)$ ② Multiple zeta values (MZV)

Formal = Symbols / Relations

And to explain the following result:

<u>Thm</u> (BIM) i) $\mathfrak{g}^f$ (and $\mathfrak{g}_0^f$) is an sl_2-algebra

$\widetilde{M}^f :=$ smallest sl_2-subalg. containing $G^f(2)$

$S^f := \operatorname{Ker}\pi|_{M^f}$ formal cusp forms

ii) $\widetilde{M}^f \cong \widetilde{M}$ as sl_2-algebras.

($\Rightarrow M^f \cong M$)

iii) $\exists$ surj. alg. hom $\pi : \mathfrak{g}^f \to \mathcal{Z}^f$.

"formal proj. to const. term"

$\operatorname{Ker}(\pi) =$ explicit.

① QMF

$k \geq 2$ even, $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, $q = e^{2\pi i \tau}$

Eisenstein series:

$$\mathbb{G}_k = \mathbb{G}_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n>0} \frac{n^{k-1} q^n}{1-q^n}$$

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k}$$

$$\overset{k \geq 4}{=} \frac{1}{2} \sum_{\lambda \in \mathbb{Z}\tau + \mathbb{Z} \setminus 0} \frac{1}{\lambda^k}$$

$$G_k := (2\pi i)^{-k} \mathbb{G}_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1-q^n}$$

Modular forms (w. $\mathbb{Q}$-coeff.) QMF

$$M = \mathbb{Q}[G_4, G_6] \overset{q\frac{d}{dq}}{\subset} \widetilde{M} = \mathbb{Q}[G_2, G_4, G_6] \circlearrowleft q\frac{d}{dq}$$

$$q\frac{d}{dq} G_2 = 5G_4 - 2G_2^2$$

$$q\frac{d}{dq} G_4 = 14 G_6 - 8 G_2 G_4$$

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An algebra A is called <u>sl_2-algebra</u> if there exist derivations $D, W, \delta \in Der(A)$, s.th.

$$[W, D] = 2D, \quad [W, \delta] = -2\delta, \quad [\delta, D] = W$$

Fact: $\widehat{M}$ is an sl_2-algebra with $D = q\frac{d}{dq}$ and $\delta(G_2) = -\frac{1}{2}$, $\delta(G_4) = \delta(G_6) = 0$, $W(G_k) = k G_k$.

Note: $M = Ker(\delta)$

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② MZV $k_1 \geq 2, k_2, \ldots, k_r \geq 1$

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \ldots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$$

$\zeta(\emptyset) = 1$.

$\mathcal{Z}$: $\mathbb{Q}$-v.s. spanned by all $\zeta(k_1, \ldots, k_r)$, $r \geq 0$

Fact: $\mathcal{Z}$ is $\mathbb{Q}$-alg.

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Quasi-shuffle product: $\mathcal{L}$: set ("letters")

$\diamond$: ass. & com. product on $\mathbb{Q}\mathcal{L}$

$1 *_\diamond w = w * 1 = w$

On $\mathbb{Q}\langle\mathcal{L}\rangle$ define the bilinear product $*_\diamond$:

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a\diamond b)(w *_\diamond v)$$

$\forall a,b \in \mathcal{L}$, w,v words (:= monic monomials in $\mathbb{Q}\langle\mathcal{L}\rangle$)

Hoffman: $(\mathbb{Q}\langle\mathcal{L}\rangle, *_\diamond)$ com. $\mathbb{Q}$-alg.

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Ex 1: $\mathcal{L}_{xy} = \{x,y\}$, $a \diamond b = 0$ $\forall a,b \in \mathcal{L}$.

$*_\diamond =: \shuffle$ <u>shuffle product</u>.

$$\mathfrak{h}^0_{\shuffle} \subset \mathfrak{h}^1_{\shuffle} \subset \mathfrak{h}_{\shuffle} := (\mathbb{Q}\langle x,y\rangle, \shuffle)$$

$\mathfrak{h}^0_{\shuffle} = \mathbb{Q} + x\mathfrak{h}y$, $\quad \mathfrak{h}^1_{\shuffle} = \mathbb{Q} + \mathfrak{h}y$

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Ex2: $\mathcal{L}_z = \{ z_k \mid k \geq 1 \}$, $z_i \diamond z_j = z_{i+j}$

$*_\diamond =: *$ <u>stuffle product</u>

$$\mathbb{Q}\langle \mathcal{L}_z \rangle \cong \mathfrak{h}^1 \rightsquigarrow \mathfrak{h}^1_* \quad \mathbb{Q}\text{-alg.}$$

$$z_{k_1} \cdots z_{k_r} \longmapsto \underbrace{x \cdots x}_{k_1 - 1} y \cdots \underbrace{x \cdots x}_{k_r - 1} y$$

$$\mathfrak{h}^1_* \supset \mathfrak{h}^0_*$$

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Fact: The map

$$\zeta : \mathfrak{h}^0_\bullet \longrightarrow \mathbb{Z}$$

$$w = z_{k_1} \cdots z_{k_r} \longmapsto \zeta(w) = \zeta(k_1, \ldots, k_r)$$

is alg. hom for $\bullet \in \{ *, ш \}$

Since $\mathfrak{h}^1_\bullet = \mathfrak{h}^0_\bullet [y]$ for $\bullet \in \{ *, ш \}$

we can extend to $\zeta^\bullet : \mathfrak{h}^1_\bullet \longrightarrow \mathbb{Z}[T]$

$$z_1 = y \longmapsto T .$$

EDS: Ideal in $\mathfrak{h}^1$ gen. by $\omega * v - \omega ш v$
for all $\omega \in \mathfrak{h}^0$, $v \in \mathfrak{h}^1$.

Conjecture: $\ker(\zeta^{\circ}) = \mathrm{EDS}$.

<u>Formal MZV</u>:

$$\mathcal{Z}^f := \mathfrak{h}^1_* \big/ \mathrm{EDS}_*$$

$\zeta^f(k_1,\dots,k_r)$: class of $z_{k_1}\cdots z_{k_r}$

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3 MES

$k_1,\dots,k_r \geq 2$

$$G_{k_1,\dots,k_r}(\tau) := \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}} \quad = G_{k_1}(\tau)$$

r=1, k_1 even

Order $\succ$:

$$m_1\tau + n_1 \succ m_2\tau + n_2 :\Longleftrightarrow m_1 > m_2$$

or $m_1 = n_1 \wedge n_1 > n_2$

$$\overset{\text{GKZ B.}}{=} \mathcal{G}(k_1,\dots,k_r) + \sum_{n>0} a_n q^n$$

$a_n \in \mathbb{Z}[2\pi i]$

Let $\mathfrak{h}^2 = \mathrm{span}\{z_{k_1}\cdots z_{k_r} \mid k_1 \dots k_r \geq 2\} \subset \mathfrak{h}^0$.

$$\Rightarrow \quad \mathbb{G} : \mathfrak{h}^2_* \longrightarrow \mathcal{O}(\mathbb{H}) \quad \text{is alg. hom.}$$
$$z_{k_1}\cdots z_{k_r} \longmapsto G_{k_1,\dots,k_r}$$

Questions:

- Extension to $\mathbb{G}^\bullet : \mathfrak{h}^1_\bullet \to \mathcal{O}(\mathbb{H})$?
- What is $\mathrm{Ker}(\mathbb{G}^\bullet)$?

$\bullet = ш$ B. Tanaka

$\bullet = *$ B.

$\rightsquigarrow$

Wish: "Formal MES" $:= \mathfrak{h}^1_* / \mathrm{Ker}(\mathbb{G}^*)$

No conj. known for this

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(4) FMES

$A = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \;\middle|\; k \geq 1, d \geq 0 \right\}$

$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ d_1 + d_2 \end{bmatrix}$, $\quad * = *_\diamond$ stuffle product

$\rightsquigarrow \ (\mathbb{Q}\langle A \rangle, *)$ $\mathbb{Q}$-alg. $\supset \mathfrak{h}'_*$

$\begin{bmatrix} k \\ 0 \end{bmatrix} \leftrightarrow z_k$

Write $\begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} := \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} \cdots \begin{bmatrix} k_r \\ d_r \end{bmatrix}$ for words

e.g. $\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1, k_2 \\ d_1, d_2 \end{bmatrix} + \begin{bmatrix} k_2, k_1 \\ d_2, d_1 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 \\ d_1 + d_2 \end{bmatrix}$

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Swap: Gen. series

$$\mathcal{A}\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{\substack{k_1, \dots, k_r \geq 1 \\ d_1, \dots, d_r \geq 0}} \begin{bmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{bmatrix} X_1^{k_1 - 1} \cdots X_r^{k_r - 1} \frac{Y_1^{d_1}}{d_1!} \cdots \frac{Y_r^{d_r}}{d_r!}.$$

Def. lin. map $\overset{\text{swap}}{\vee}$ $\sigma : \mathbb{Q}\langle A \rangle \longrightarrow \mathbb{Q}\langle A \rangle$ by

$$\sigma\left(\mathcal{A}\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix}\right) := \mathcal{A}\begin{pmatrix} Y_1 + \dots + Y_r, & Y_1 + \dots + Y_{r-1}, & \dots, & Y_1 \\ X_r, & X_{r-1} - X_r, & \dots, & X_1 - X_2 \end{pmatrix}$$

ex: $\sigma\left(\begin{bmatrix} k \\ d \end{bmatrix}\right) = \dfrac{d!}{(k-1)!} \begin{bmatrix} d+1 \\ k-1 \end{bmatrix}$

Algebra of FMES:

$$\mathcal{G}^f := (\mathbb{Q}\langle A\rangle, *) \big/ I$$

I: ideal gen. by. $\delta(w) - w$ for all $w \in \mathbb{Q}\langle A\rangle$.

$G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$: class of $\begin{bmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{bmatrix}$

$k_1,\dots,k_r \geq 1$, $d_1,\dots,d_r \geq 0$

Set $G^f(k_1,\dots,k_r) := G^f\binom{k_1,\dots,k_r}{0,\dots,0}$

$$\mathcal{G}^f_0 := \mathrm{span}_{\mathbb{Q}}\{ G^f(k_1,\dots,k_r) \mid r \geq 0, k_1,\dots,k_r \geq 1\}$$

$$\left(\mathcal{E}^f := \mathrm{span}\{ G^f(k_1,\dots,k_r) \mid k_j \geq 2\} \right)$$

<u>Conj</u>: $\mathcal{G}^f = \mathcal{G}^f_0$, $(\mathcal{E}^f \cong \mathcal{E})$

Open problem: Find R with $\mathcal{G}^f \cong \mathfrak{h}^1_x \big/ R$

(Claim: $G^f(k_1,\dots,k_r)$ are the "correct" formal analogue of MES.)

Derivations on $\mathfrak{g}^f$:

$$D: \mathfrak{g}^f \longrightarrow \mathfrak{g}^f$$
$$G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \longmapsto \sum_{j=1}^{r} k_j\, G^f\binom{k_1,\dots,k_j+1,\dots,k_r}{d_1,\dots,d_j+1,\dots,d_r}$$

$$W: \mathfrak{g}^f \longrightarrow \mathfrak{g}^f$$
$$G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \longmapsto (k_1+\dots+k_r+d_1+\dots+d_r)\, G^f\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$$

Thm (BIM) i) There exists an explicit $\delta \in \mathrm{Der}(\mathfrak{g}^f)_1$ s.th. $\mathfrak{g}^f$ is an $\mathfrak{sl}_2$-algebra.

ii) $\mathfrak{g}_0^f$ is an $\mathfrak{sl}_2$-subalgebra

iii) $\widetilde{\mathcal{M}}^f = \mathbb{Q}[G^f(2), G^f(4), G^f(6)] \cong \widetilde{\mathcal{M}}$.

$\cup$

$\mathcal{M}^f := \operatorname{Ker} \delta|_{\widetilde{\mathcal{M}}^f} \cong \mathcal{M}$

$\cup$

$\mathcal{S}^f := \operatorname{Ker} \pi|_{\mathcal{M}^f} \cong \mathcal{S}$

Idea of Proof:

i) Study derivations on $(\mathbb{Q}\langle A\rangle, *)$

δ = sum of 5 derivations $\delta_1, \dots, \delta_5$

s.th. δ is $\mathfrak{G}$-equivariant

ii) Use swap invariance to show that for $w \in \mathfrak{h}'$

$$D\, G^f(w) = G^f(z_2 * w - z_2 ш w).$$

iii) Use alg. hom. $G: \mathcal{G}^f \longrightarrow \mathbb{Q}[[q]]$ (B.-Burmester)

$G^f(k) \longmapsto G_k$

Let N be the ideal in $\mathcal{G}^f$ gen. by all FMES <u>not</u> of the form

$$G\begin{pmatrix} 1, \dots, 1, & k_1, \dots, k_r \\ d_1, \dots, d_s, & 0, \dots, 0 \end{pmatrix}. \qquad s, r \geq 0$$

<u>Thm</u> (BIM)

$$\pi: \mathcal{G}^f \longrightarrow \mathcal{G}^f / N \cong \mathcal{Z}^f$$