A combinatorial approach to classical modular forms inspired by multiple zeta values

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Eisenstein series identities

Definition

For even $k > 2$ the Eisenstein series of weight k is defined by

$$G_k(\tau) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(m\tau+n)^k} = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n \,,$$

where $\tau \in \{x+iy \in \mathbb{C} \mid y > 0\}$, $q = \exp(2\pi i\tau)$ and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

- The Eisenstein series G_k is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$.
- The spaces of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ and their dimensions are well understood.
- For even k the Riemann zeta values $\zeta(k)$ are known to be rational multiples of π^k, e.g.

$$\zeta(2) = \frac{\pi^2}{6}\,, \quad \zeta(4) = \frac{\pi^4}{90}\,, \quad \zeta(6) = \frac{\pi^6}{945}\,, \quad \zeta(8) = \frac{\pi^8}{9450}\,.$$

Eisenstein series identities

With all this knowledge the following Propositon ist absolutely trivial:

Proposition

We have the following identity

$$G_4^2(\tau) = \frac{7}{6} G_8(\tau) \,.$$

Proof:

- G_4^2 and G_8 are modular forms of weight 8
- The space of weight 8 modular forms has dimension 1.
- Their Fourier expansion both have $\zeta(4)^2 = \frac{7}{6}\zeta(8)$ as their constant term.

$\square$

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But how do you prove this without knowing modular forms and $\zeta(4)^2 = \frac{7}{6}\zeta(8)$?

Aim of this talk

In this talk I try to present a purely combinatorial way to prove such relations.

For this we need...

- **Multiple zeta values (MZV)** - multiple version of the Riemann zeta values
- **Double shuffle relations** - a toolbox to prove relations between MZV
- **Multiple Eisenstein series** - multiple version of the Eisenstein series

Multiple zeta values

Definition

For natural numbers $s_1 \geq 2, s_2, ..., s_l \geq 1$, the multiple zeta value (MZV) of weight $k = s_1 + \cdots + s_l$ and length l is defined by

$$\zeta(s_1, \ldots, s_l) := \sum_{n_1 > n_2 > \cdots > n_l > 0} \frac{1}{n_1^{s_1} \ldots n_l^{s_l}} .$$

By $\mathcal{MZ}_k$ we denote the space spanned by all MZV of weight k and by $\mathcal{MZ}$ the space spanned by all MZV.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (**harmonic product**)
- MZV can be expressed as iterated integrals. This gives another way (**shuffle product**) to express the product of two MZV as a linear combination of MZV.
- These two products give a number of $\mathbb{Q}$-relations (**double shuffle relations**) between MZV.

Multiple zeta values - harmonic product

In the smallest length the harmonic product reads

$$\begin{aligned}
\zeta(s_1) \cdot \zeta(s_2) &= \sum_{n_1>0} \frac{1}{n_1^{s_1}} \sum_{n_2>0} \frac{1}{n_2^{s_2}} \\
&= \sum_{n_1>n_2>0} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_2>n_1>0} \frac{1}{n_1^{s_1} n_2^{s_2}} + \sum_{n_1=n_2>0} \frac{1}{n_1^{s_1+s_2}} \\
&= \zeta(s_1, s_2) + \zeta(s_2, s_1) + \zeta(s_1 + s_2).
\end{aligned}$$

Multiple zeta values - harmonic product

In the smallest length the harmonic product reads

$$\begin{aligned}\zeta(s_1)\cdot\zeta(s_2) &= \sum_{n_1>0}\frac{1}{n_1^{s_1}}\sum_{n_2>0}\frac{1}{n_2^{s_2}} \\ &= \sum_{n_1>n_2>0}\frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_2>n_1>0}\frac{1}{n_1^{s_1}n_2^{s_2}} + \sum_{n_1=n_2>0}\frac{1}{n_1^{s_1+s_2}} \\ &= \zeta(s_1,s_2)+\zeta(s_2,s_1)+\zeta(s_1+s_2)\,.\end{aligned}$$

For length 1 times length 2 the same argument gives

$$\begin{aligned}\zeta(s_1)\cdot\zeta(s_2,s_3) = \zeta(s_1,s_2,s_3)+\zeta(s_2,s_1,s_3)+\zeta(s_2,s_3,s_1) \\ +\zeta(s_1+s_2,s_3)+\zeta(s_2,s_1+s_3)\,.\end{aligned}$$

Multiple zeta values - shuffle product

Multiple zeta values can also be written as iterated integrals. For example

$$\zeta(2,3) = \int_{1>t_1>t_2>t_3>t_4>t_5>0} \underbrace{R(t_1)B(t_2)}_{2}\underbrace{R(t_3)R(t_4)B(t_5)}_{3},$$

with the differential forms $R(t) = \frac{dt}{t}$ and $B(t) = \frac{dt}{1-t}$.

Multiple zeta values - shuffle product

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with the differential forms $R(t) = \frac{dt}{t}$ and $B(t) = \frac{dt}{1-t}$.

Multiplying two such integrals results in the sum of all possible shuffles of the integrants

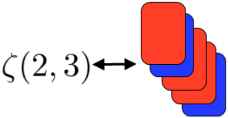
$$\zeta(2)\cdot\zeta(3) = \int_{1>t_1>t_2>0} R(t_1)B(t_2) \cdot \int_{1>u_1>u_2>u_3>0} R(u_1)R(u_2)B(u_3)$$

$$= \int_{1>t_1>t_2>u_1>u_2>u_3>0} \ldots + \int_{1>t_1>u_1>t_2>u_2>u_3>0} \ldots + \ldots$$

Multiple zeta values - shuffle product

Suppose we have two types of cards (red and blue).

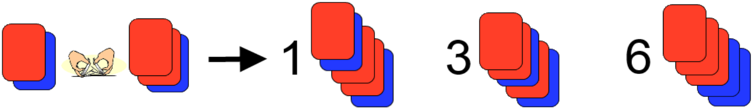
- MZV correspond to a deck of these cards



- Multiplication of MZV corresponds to shuffling two of these decks (+counting multiplicities)

Multiple zeta values - shuffle product - example

For example the product $\zeta(2) \cdot \zeta(3)$ can be evaluated as



$$\zeta(2) \cdot \zeta(3) \quad = \quad \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$$

Multiple zeta-values - double shuffle relations

These two representations for the product give a large family of linear relations between MZV.

$$\zeta(3,2) + 3\zeta(2,3) + 6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(2,3) + 6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(2,1) = \zeta(3).$$

These follow from the "extended double shuffle relations" where one use the same combinatorics as above for "$\zeta(1)\cdot\zeta(2)$" in a formal setting.

Multiple zeta values - double shuffle relations - example

Now we have enough tools to prove $\zeta(4)^2 = \frac{7}{6}\zeta(8)$.

With the harmonic (h) and shuffle (s) product we obtain

$$\zeta(4)\cdot\zeta(4) \overset{\text{h}}{=} 2\zeta(4,4) + \zeta(8)\,, \tag{1}$$

$$\zeta(4)\cdot\zeta(4) \overset{\text{s}}{=} 2\zeta(4,4) + 8\zeta(5,3) + 20\zeta(6,2) + 40\zeta(7,1) \tag{2}$$

$$\zeta(3)\cdot\zeta(5) \overset{\text{h}}{=} \zeta(3,5) + \zeta(5,3) + \zeta(8)\,, \tag{3}$$

$$\zeta(3)\cdot\zeta(5) \overset{\text{s}}{=} \zeta(3,5) + 3\zeta(4,4) + 7\zeta(5,3) + 15\zeta(6,2) + 30\zeta(7,1) \tag{4}$$

From which we deduce

$$\zeta(4)^2 = 2\zeta(4,4) + \zeta(8) = \underbrace{\frac{2}{3}\left((4)-(3)\right)}_{=0} - \underbrace{\frac{1}{2}\left((2)-(1)\right)}_{=0} + \frac{7}{6}\zeta(8)\,.$$

Back to Eisenstein series..

But how can we prove $G_4(\tau)^2 = \frac{7}{6} G_8(\tau)$?

- Introduce multiple Eisenstein series.
- Show that product of two multiple Eisenstein series can also be express by the harmonic and the shuffle product.
- With this one can use the exact same proof as before by replacing ζ with G.

Multiple Eisenstein series

Definition

For $s_1, \dots, s_l \geq 2$ we define the **multiple Eisenstein series** of weight $k = s_1 + \dots + s_l$ and length l by

$$G_{s_1,\dots,s_l}(\tau) := \sideset{}{'}\sum_{\substack{\lambda_1 \succ \dots \succ \lambda_l \succ 0 \\ \lambda_i \in \Lambda_\tau}} \frac{1}{\lambda_1^{s_1} \dots \lambda_l^{s_l}} \,,$$

where $\lambda_i \in \mathbb{Z}\tau + \mathbb{Z}$ are lattice points and the order $\succ$ on $\mathbb{Z} + \mathbb{Z}\tau$ is given by

$$m_1\tau + n_1 \succ m_2\tau + n_2 :\Leftrightarrow (m_1 > m_2 \vee (m_1 = m_2 \wedge n_1 > n_2)) \;.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the harmonic product, i.e. as for MZV we have

$$G_2(\tau) \cdot G_3(\tau) = G_{2,3}(\tau) + G_{3,2}(\tau) + G_5(\tau) \,.$$

Multiple Eisenstein series - shuffle product?

What about the shuffle product?

Remember for MZV we have

$$\zeta(2) \cdot \zeta(3) = \zeta(3,2) + 3\zeta(2,3) + 6\zeta(4,1)\,.$$

This equation does not make sense for multiple Eisenstein series if we replace ζ by G since there is no definition of $G_{4,1}$.

Question

What is a good definition of $G_{s_1,\dots,s_l}$ for $s_1 \geq 2, s_2, \dots, s_l \geq 1$ such these series also "fulfill" the shuffle product ?

Multiple Eisenstein series - shuffle regularization

Theorem (B., K. Tasaka 2014)

For all $s_1, \ldots, s_l \geq 1$ there exist shuffle regularized multiple Eisenstein series $G^{ш}_{s_1,\ldots,s_l}$ with the following properties:

- They are holomorphic functions on the upper half plane having a Fourier expansion with the multiple zeta values as the constant term.
- They "fulfill" the shuffle product.
- For integers $s_1, \ldots, s_l \geq 2$ they equal the multiple Eisenstein series

$$G^{ш}_{s_1,\ldots,s_l}(\tau) = G_{s_1,\ldots,s_l}(\tau)$$

and therefore they fulfill the harmonic product in these cases.

Proof sketch: Uses a beautiful connection of the Fourier expansion of multiple Eisenstein series to the coproduct of formal iterated integrals.

Double shuffle relations for multiple Eisenstein series

The Theorem enables one to use the double shuffle relations for products of multiple Eisenstein series $G_{s_1,\dots,s_l} \cdot G_{r_1,\dots,r_m}$ whenever $s_1,\dots,s_l,r_1,\dots,r_m \geq 2$.

Proposition

We have the following identity

$$G_4^2(\tau) = \frac{7}{6} G_8(\tau)\,.$$

Alternative proof: Use the double shuffle relations for $G_4 \cdot G_4$ and $G_3 \cdot G_5$. □

All algebraic relations between Eisenstein series can be proven this way.

Double shuffle relations for multiple Eisenstein series

- There are relations between MZV, which can be proven by double shuffle but which are not true for Eisenstein series.
- For example the relation

$$\zeta(6)^2 = \frac{715}{691}\zeta(12)$$

can be proven by using the double shuffle relations. But this relation is not true for Eisenstein series, because there are cusp forms in weight 12, i.e. for some $c \in \mathbb{R}$

$$G_6(\tau)^2 = \frac{715}{691}G_{12}(\tau) + c \cdot \Delta \,.$$

But why?

"So you study these things just to give alternative proofs for easy & well-known results?"

No....

- In the theory of multiple zeta values modular forms appear in several ways.
- There are relations between multiple zeta values which "come from cusp forms".
- Understanding the failure of the double shuffle relations for multiple Eisenstein series explain these relations.
- This failure is still not well understood.

Summary

- Multiple zeta values (MZV) are multiple version of the Riemann zeta values.
- $\mathbb{Q}$-linear relation between these real numbers can be proven by expressing the product of two MZV in two different ways. (harmonic & shuffle product)
- There also exist multiple version of the classical Eisenstein series given by multiple Eisenstein series.
- Multiple Eisenstein series also fulfill "some but not all" of the double shuffle relations.
- This "some but not all" is crucial to understand the modular aspect of MZV, but still not well understood so far.

Thank you for your attention!