Multiple harmonic q-series at roots of unity and their connection to finite & symmetrized multiple zeta values

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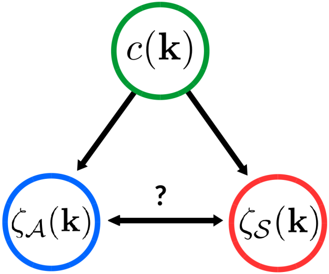

joint work with Y. Takeyama and K. Tasaka (arXiv:1707.05008)

AG-Seminar Algebra, TU Darmstadt, 06.02.2018

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Multiple harmonic q-series

$$q = e^{2\pi i \tau}$$

Classical case

$\tau \in \mathbb{H}$ $\qquad \mathbf{k} = (k_1, \ldots, k_r)$

Modular forms $\subset g(\mathbf{k}) \subset \mathcal{O}(\mathbb{H})$

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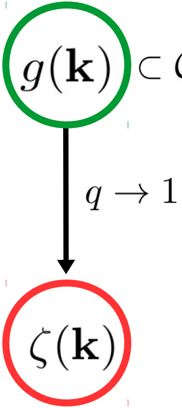
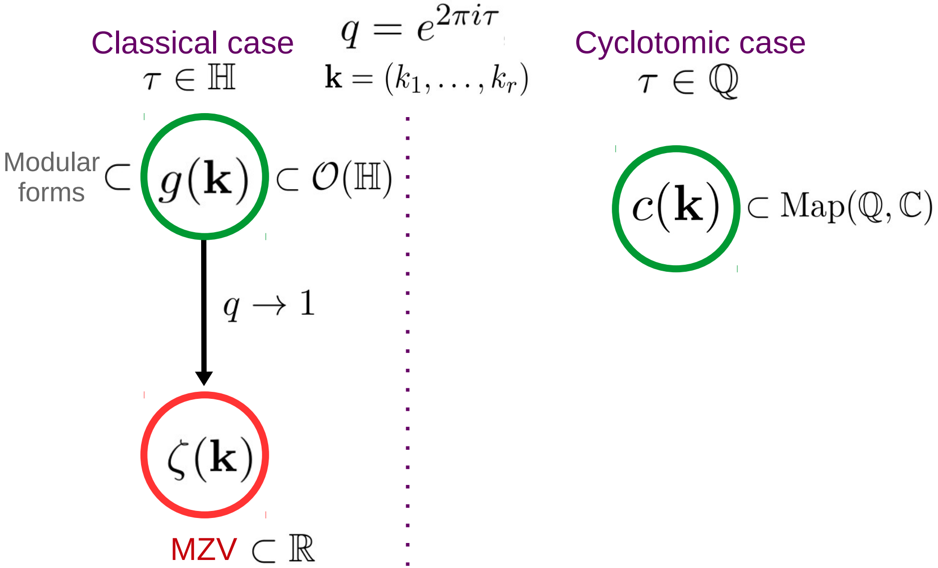


MZV $\subset \mathbb{R}$

MZV = Multiple zeta values

Multiple harmonic q-series



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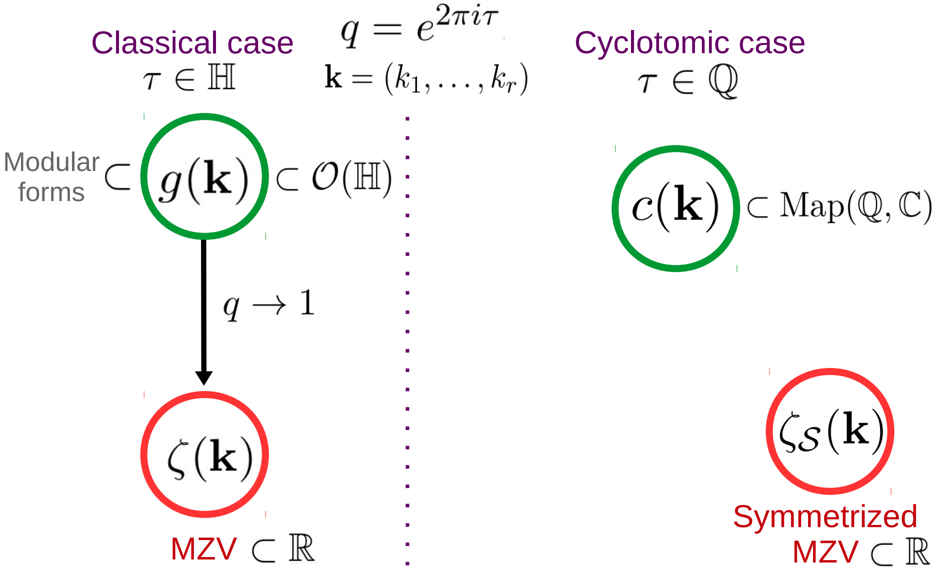

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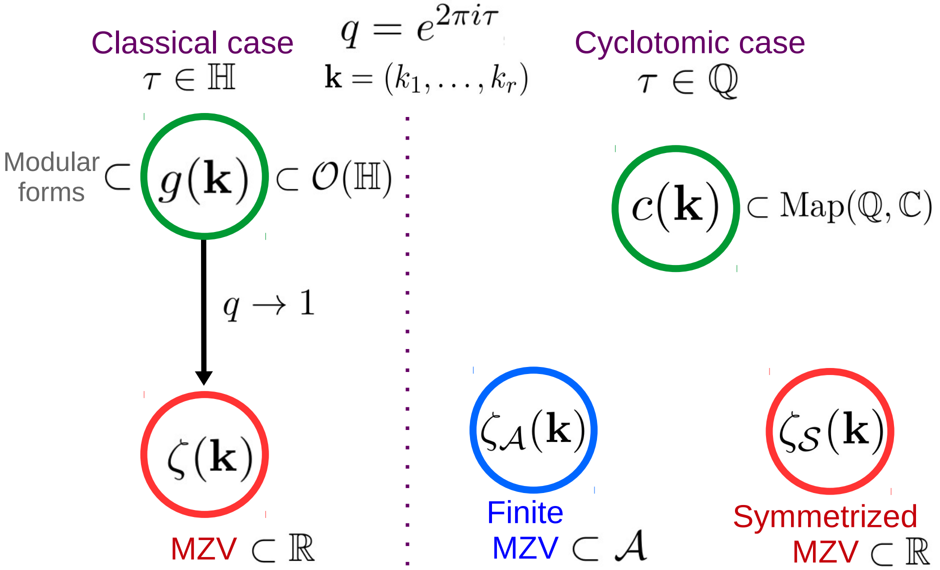

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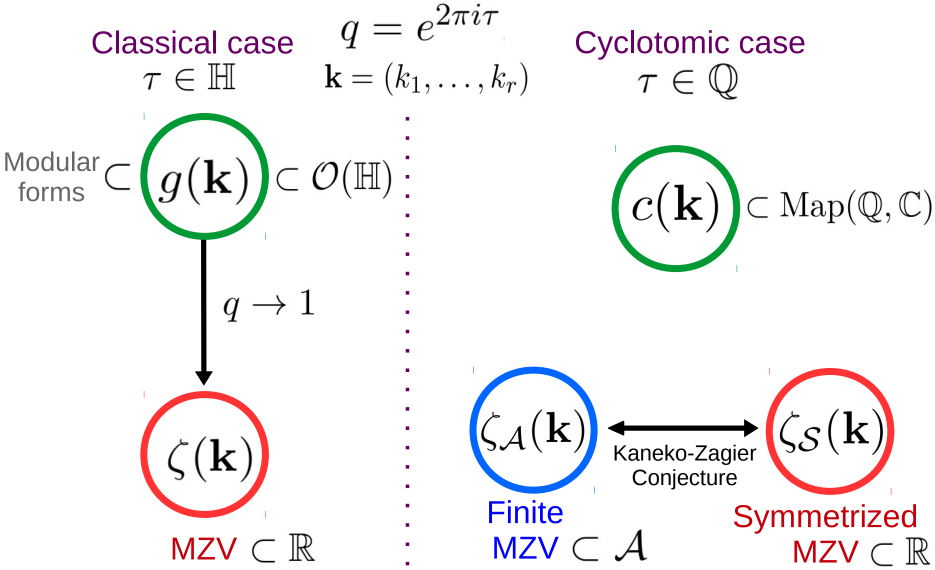
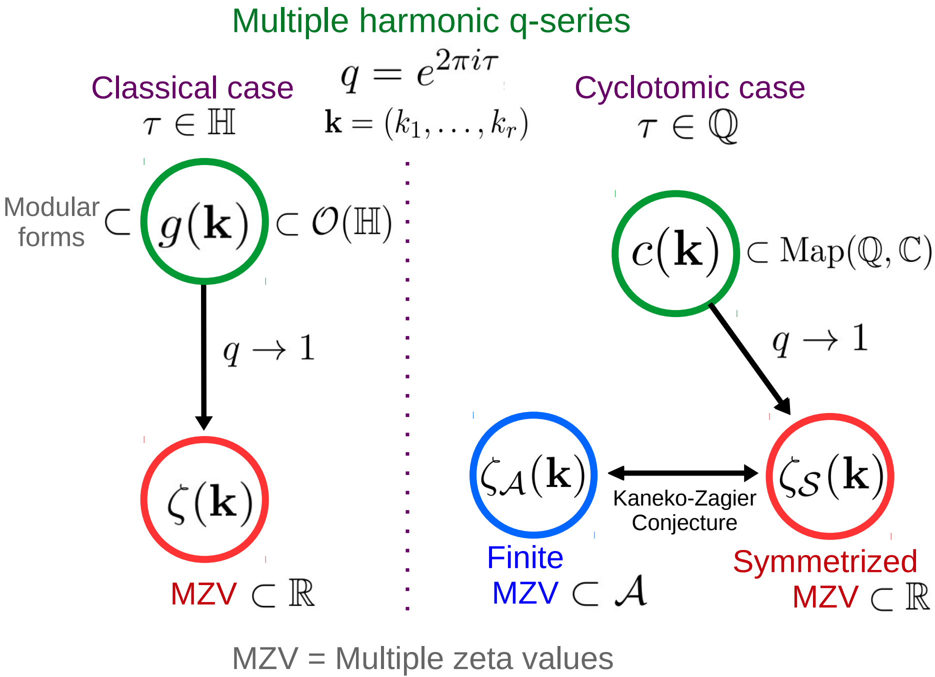
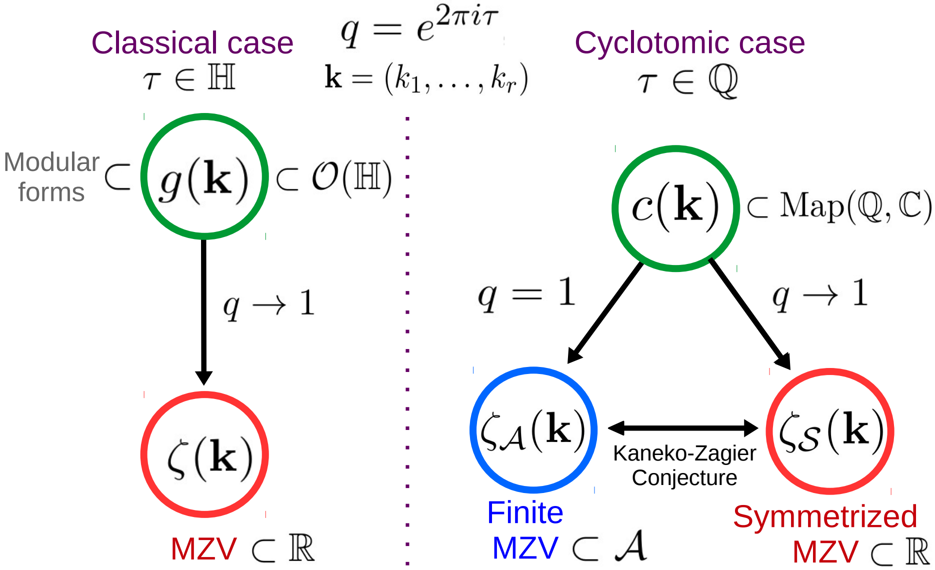


MZV = Multiple zeta values



Multiple harmonic q-series
$q = e^{2\pi i\tau}$
Classical case
$\tau \in \mathbb{H}$
$\mathbf{k} = (k_1, \ldots, k_r)$
Cyclotomic case
$\tau \in \mathbb{Q}$
Modular forms $\subset$ $g(\mathbf{k})$ $\subset \mathcal{O}(\mathbb{H})$
$c(\mathbf{k})$ $\subset \mathrm{Map}(\mathbb{Q}, \mathbb{C})$
$q \to 1$
$q \to 1$
$\zeta(\mathbf{k})$
$\zeta_{\mathcal{A}}(\mathbf{k})$
Kaneko-Zagier Conjecture
$\zeta_{\mathcal{S}}(\mathbf{k})$
MZV $\subset \mathbb{R}$
Finite MZV $\subset \mathcal{A}$
Symmetrized MZV $\subset \mathbb{R}$
MZV = Multiple zeta values

Multiple harmonic q-series



MZV = Multiple zeta values

- Indexset: $\mathbf{k} = (k_1, \dots, k_r)$, $k_1, \dots, k_r \geq 1$,
- $\mathbf{k}$ **admissible** $:\Leftrightarrow k_1 \geq 2$ or $\mathbf{k} = \emptyset$ $(r = 0)$,
- **weight**: $\mathrm{wt}(\mathbf{k}) = k_1 + \dots + k_r$, **depth**: r.

Definition

- For $\mathbf{k}$ admissible define the **multiple zeta value** (MZV)

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}$$

and set $\zeta(\emptyset) = 1$.

- For the space spanned by all MZV we write

$$\mathcal{Z} = \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible} \rangle_{\mathbb{Q}}$$

and

$$\mathcal{Z}_k = \langle \zeta(\mathbf{k}) \mid \mathbf{k} \text{ admissible}, \mathrm{wt}(\mathbf{k}) = k \rangle_{\mathbb{Q}} .$$

For $k_1, k_2 \geq 2$ we have

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m_1>0}\frac{1}{m_1^{k_1}}\sum_{m_2>0}\frac{1}{m_2^{k_2}}\\ &= \left(\sum_{m_1>m_2>0}+\sum_{m_1>m_2>0}+\sum_{m_1=m_2>0}\right)\frac{1}{m_1^{k_1}m_2^{k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

This works for arbitrary depths, for example

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2,k_3) = \zeta(k_1,k_2,k_3)+\zeta(k_2,k_1,k_3)+\zeta(k_2,k_3,k_1)\\ +\zeta(k_1+k_2,k_3)+\zeta(k_2,k_1+k_3)\,.\end{aligned}$$

In particular $\mathcal{Z}$ is a $\mathbb{Q}$-algebra.

Additionally MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.

Example

$$\zeta(2,3)+3\zeta(3,2)+6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(2,1)=\zeta(3).$$

These type of relations are called **extended double shuffle relations**.

MZV Conjectures

- The extended double shuffle relations give all linear relations among MZV and

$$\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k \,,$$

  i.e. there are no relations between MZV of different weight.

- (Zagier) The dimension of the spaces $\mathcal{Z}_k$ is given by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k \, X^k = \frac{1}{1 - X^2 - X^3} \,.$$

- (Hoffman) The following set gives a basis of $\mathcal{Z}$

$$\{\zeta(k_1,\dots,k_r) \mid r \geq 0, k_1,\dots,k_r \in \{2,3\}\} \,.$$

Theorem (Goncharov, Terasoma)

We have $\dim_{\mathbb{Q}} \mathcal{Z}_k \le d_k$, where $\sum_{k\ge 0} d_k\, X^k := (1 - X^2 - X^3)^{-1}$.

k	0	1	2	3	4	5	6	7	8	9	10	11
$\#\mathbf{k}$ adm., $\mathrm{wt} = k$	1	0	1	2	4	8	16	32	64	128	256	512
d_k	1	0	1	1	1	2	2	3	4	5	7	9

Theorem (Brown, 2011)

Every MZV can be written as a linear combination of $\zeta(k_1, \ldots, k_r)$ with $k_j \in \{2, 3\}$.

Example

$$\zeta(4) = \frac{4}{3}\zeta(2,2)\,, \qquad \zeta(5) = \frac{6}{5}\zeta(2,3) + \frac{4}{5}\zeta(3,2)\,,$$

$$\zeta(4,1) = \frac{1}{5}\zeta(2,3) - \frac{1}{5}\zeta(3,2)\,, \qquad \zeta(6) = \frac{16}{3}\zeta(2,2,2)\,.$$

Theorem (Gangl-Kaneko-Zagier, 2006)

Modular forms of weight k "give" relations between $\zeta(r,s)$ and $\zeta(k)$ with $k = r+s$ and r, s odd.

There are explicit formulas for these relation using period polynomials.

Example

- Each Eisenstein series in weight k corresponds to the relation

$$\zeta(3,k-3) + \zeta(5,k-5) + \cdots + \zeta(k-3,3) + \zeta(k-1,1) = \frac{1}{4}\zeta(k)\,.$$

- The cusp form Δ in weight 12 gives

$$168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3) = \frac{5197}{691}\zeta(12)\,.$$

We will see later how this relation can be related to Δ.

Definition

For $k_1, \ldots, k_r \geq 1$ there exists a unique $\zeta(k_1, \ldots, k_r; T) \in \mathcal{Z}[T]$ with

- $\zeta(1; T) = T$,
- For $k_1 \geq 2$ it is $\zeta(k_1, \ldots, k_r; T) = \zeta(k_1, \ldots, k_r)$,
- Their product can be expressed by the harmonic product formula.

Example

Since

$$\zeta(1; T) \cdot \zeta(2; T) = \zeta(1, 2; T) + \zeta(2, 1; T) + \zeta(3; T)$$

we have

$$\zeta(1, 2; T) = \zeta(2)T - \zeta(2, 1) - \zeta(3)\,.$$

In general we have for $\mathbf{k}$ admissible: $\zeta(\underbrace{1, \ldots, 1}_{m}, \mathbf{k}; T) = \zeta(\mathbf{k})\frac{T^m}{m!} + \ldots.$

Definition

For an indexset $\mathbf{k} = (k_1, \ldots, k_r)$ define the **symmetrized multiple zeta value** by

$$\zeta_{\mathcal{S}}(\mathbf{k}) = \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta(k_a, k_{a-1}, \ldots, k_1; T) \zeta(k_{a+1}, k_{a+2}, \ldots, k_r; T) \,.$$

- One can check that the definition of $\zeta_{\mathcal{S}}$ is independent of T.
- The product of two SMZV can again be expressed by the harmonic product, e.g.

$$\zeta_{\mathcal{S}}(k_1) \cdot \zeta_{\mathcal{S}}(k_2) = \zeta_{\mathcal{S}}(k_1, k_2) + \zeta_{\mathcal{S}}(k_2, k_1) + \zeta_{\mathcal{S}}(k_1 + k_2) \,.$$

In depth $r = 1$ we have for $k \geq 1$

$$\zeta_{\mathcal{S}}(k) = \zeta(k;T) + (-1)^k \zeta(k;T) = \begin{cases} 2\zeta(k) & , k \text{ is even} \\ 0 & , k \text{ is odd} \end{cases} .$$

Question: Do we get all MZV?

Theorem (Yasuda, 2014)

We have $\mathcal{Z} = \langle \zeta_{\mathcal{S}}(\mathbf{k}) \rangle_{\mathbb{Q}}$.

Relations between MZV give relation between Symmetrized MZV:

Example

$$\zeta(5) - 2\zeta(2,3) + 4\zeta(4,1) = 0$$

$$\Longleftrightarrow$$

$$\zeta_{\mathcal{S}}(4,1) - \zeta_{\mathcal{S}}(1,4) + \zeta_{\mathcal{S}}(3,2) = 0$$

Definition

For an indexset $\mathbf{k} = (k_1, \ldots, k_r)$ the **finite multiple zeta value** is defined by

$$\zeta_{\mathcal{A}}(\mathbf{k}) = \left(\sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_{p \text{ prime}} \in \mathcal{A}\,,$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = \prod_{p \text{ prime}} \mathbb{F}_p \Big/ \bigoplus_{p \text{ prime}} \mathbb{F}_p \,.$$

$(\ \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \ )$

We have an embedding $\mathbb{Q} \overset{i}{\hookrightarrow} \mathcal{Z}^{\mathcal{A}}$, since for $\frac{a}{b} \in \mathbb{Q}$ we can get a solution x_p of

$$b\, x_p - a \equiv 0 \mod p$$

for all but finitely many p. Set $x_p = 0$ if it does not exists and define

$$i\left(\frac{a}{b}\right) = (x_2, x_3, x_5, x_7, \dots) \in \mathcal{A} = \prod_{p \text{ prime}} \mathbb{F}_p \Big/ \bigoplus_{p \text{ prime}} \mathbb{F}_p\,.$$

$\Longrightarrow$ **$\mathcal{A}$ is a $\mathbb{Q}$-algebra.**

Example

$$i\left(\frac{3}{10}\right) = (0, 0, 0, 1, 8, 12, 2, 6, 21, \dots)\,.$$

For the space spanned by all FMZVs we write

$$\mathcal{Z}^{\mathcal{A}} = \langle \zeta_{\mathcal{A}}(\mathbf{k}) \rangle_{\mathbb{Q}} \qquad \text{and} \qquad \mathcal{Z}^{\mathcal{A}}_k = \langle \zeta_{\mathcal{A}}(\mathbf{k}) \mid \mathrm{wt}(\mathbf{k}) = k \rangle_{\mathbb{Q}} \,.$$

Finite MZV satisfy the same harmonic product formula, i.e. for example

$$\zeta_{\mathcal{A}}(k_1) \cdot \zeta_{\mathcal{A}}(k_2) = \zeta_{\mathcal{A}}(k_1, k_2) + \zeta_{\mathcal{A}}(k_2, k_1) + \zeta_{\mathcal{A}}(k_1 + k_2)$$

and therefore $\mathcal{Z}^{\mathcal{A}}$ is a $\mathbb{Q}$-algebra.

Proposition

- Depth 1: For $k \geq 1$ we have $\zeta_{\mathcal{A}}(k) = 0$.
- Depth 2: For $k_1, k_2 \geq 1$ we have

$$\zeta_{\mathcal{A}}(k_1, k_2) = \left((-1)^{k_1} \binom{k_1 + k_2}{k_2} \frac{B_{p-k_1-k_2}}{k_1 + k_2} \right)_{p \text{ prime}} .$$

- Clearly $\zeta_{\mathcal{A}}(k_1, k_2) = 0$ if $k_1 + k_2$ is even.
- It is expected, that $\zeta_{\mathcal{A}}(k_1, k_2) \neq 0$ if $k_1 + k_2$ is odd.
- We do not know an example for $\mathbf{k} \neq \emptyset$, for which we can prove $\zeta_{\mathcal{A}}(\mathbf{k}) \neq 0$.

In their work Kaneko and Zagier prove several linear relations among Finite MZV.

Example

$$\zeta_{\mathcal{A}}(4,1) - \zeta_{\mathcal{A}}(1,4) + \zeta_{\mathcal{A}}(3,2) = 0$$

They also made the following observation

Observation (Kaneko, Zagier)

The number of relations between $\zeta_{\mathcal{A}}(2a,1,2b,1)$ seems to correspond to cusp forms in weight $2(a+b+1)$.

For example in weight 12 the first relation of this type is given by

$$16\zeta_{\mathcal{A}}(2,1,8,1) + 9\zeta_{\mathcal{A}}(4,1,6,1) + 18\zeta_{\mathcal{A}}(6,1,4,1) - 2\zeta_{\mathcal{A}}(8,1,2,1) = 0\,.$$

There are no proven results on this observation (as far as I know).

Conjecture (Kaneko-Zagier)

- We have an $\mathbb{Q}$-algebra isomorphism

$$\begin{aligned}\varphi_{KZ} : \mathcal{Z}^{\mathcal{A}} &\longrightarrow \mathcal{Z}/\pi^2\mathcal{Z}\\ \zeta_{\mathcal{A}}(\mathbf{k}) &\longmapsto \zeta_{\mathcal{S}}(\mathbf{k}) \mod \pi^2\mathcal{Z}\,.\end{aligned}$$

- The dimension of $\mathcal{Z}_k^{\mathcal{A}}$ is given by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k^{\mathcal{A}} X^k = \frac{1-X^2}{1-X^2-X^3}\,.$$

- We do not even know if the map φ_{KZ} is well-defined.
- In contrast to MZV, there is no explicit conjectured basis for $\mathcal{Z}^{\mathcal{A}}$.

③ Multiple harmonic q-series - general form

A **multiple harmonic q-series** is a sum of the form

$$\sum_{m_1>\cdots>m_r>0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}},$$

with $Q_j(x) \in \mathbb{Q}[x]$ and $k_j \in \mathbb{Z}_{\geq 1}$.

With $q = e^{2\pi i\tau}$, we will consider:

- "classical case": $\tau \in \mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}, \ |q| < 1.$
- "cyclotomic case": $\tau \in \mathbb{Q}, \ q$: root of unity.

Eisenstein series of weight $k \geq 2$:

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n = \zeta(k) + (2\pi i)^k \sum_{m>0} \frac{P_k(q^m)}{(1-q^m)^k}$$

$P_k(x)$: **Eulerian polynomials**, defined for $k \geq 1$ by the Polylogarithm

$$\frac{P_k(x)}{(1-x)^k} := \frac{1}{(k-1)!} \mathrm{Li}_{1-k}(x) = \frac{1}{(k-1)!} \sum_{a>0} a^{k-1} x^a \,.$$

We have $P_k(1) = 1$ and $P_k(0) = 0$ for all $k \geq 1$.

Example

$$P_1(x) = P_2(x) = x\,, \qquad P_3(x) = \frac{1}{2}x^2 + \frac{1}{2}x$$

$$P_4(x) = \frac{1}{6}x^3 + \frac{2}{3}x^2 + \frac{1}{6}x\,.$$

Definition

- For $|q| < 1$ and any indexset $\mathbf{k} = (k_1, \ldots, k_r)$ define

$$g(\mathbf{k}; q) = g(k_1, \ldots, k_r; q) = \sum_{m_1 > \cdots > m_r > 0} \frac{P_{k_1}(q^{m_1})}{(1 - q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1 - q^{m_r})^{k_r}}$$

  and set $g(\emptyset; q) = 1$.
- If we view this as a function (in τ) from $\mathbb{H}$ to $\mathbb{C}$ we just write $g(\mathbf{k})$.
- For the space spanned by all these functions we write

$$\mathcal{G} = \langle g(\mathbf{k}) \mid \mathbf{k} \text{ admissible} \rangle_{\mathbb{Q}} \subset \mathcal{O}(\mathbb{H}) \,.$$

  and

$$\mathcal{G}_k = \langle g(\mathbf{k}) \mid \mathbf{k} \text{ admissible}, \operatorname{wt}(\mathbf{k}) \leq k \rangle_{\mathbb{Q}} \,.$$

Proposition

The space $\mathcal{G}$ is a $\mathbb{Q}$-algebra.

Denote for $n \geq 1$ the normalized Eisenstein series by

$$\widetilde{G}_{2n}(\tau) := (2\pi i)^{-2n} G_{2n}(\tau) = \frac{1}{2} \frac{B_{2n}}{(2n)!} + g(2n; q) \in \mathcal{G}\,.$$

The space of **quasi-modular forms** $\mathbb{Q}[\widetilde{G}_2, \widetilde{G}_4, \widetilde{G}_6] \subset \mathcal{G}$ is a sub algebra of $\mathcal{G}$.

Theorem (B.-Kühn, 2013)

The space $\mathcal{G}$ is closed under the operator $q\frac{d}{dq}$, and $q\frac{d}{dq}\mathcal{G}_{k-2} \subset \mathcal{G}_k$.

Example

$$q\frac{d}{dq}g(1) = g(3) - g(2,1) + \frac{1}{2}g(2).$$

The functions $g(\mathbf{k})$ can be seen as q-analogues of MZV:

Proposition

For $\mathbf{k}$ admissible and $2 \leq \mathrm{wt}(\mathbf{k}) \leq k$ we have

$$\lim_{q \to 1} (1-q)^k g(\mathbf{k}; q) = \begin{cases} \zeta(\mathbf{k}) & , \mathrm{wt}(\mathbf{k}) = k \\ 0 & , \mathrm{wt}(\mathbf{k}) < k \end{cases} .$$

We get a surjective linear map

$$\begin{aligned} \varphi_k : \mathcal{G}_k &\longrightarrow \mathcal{Z}_k \\ g(\mathbf{k}) &\longmapsto \lim_{q \to 1} (1-q)^k g(\mathbf{k}; q) \,. \end{aligned}$$

Theorem (B.-Kühn, 2013)

We have the following elements in the kernel of φ_k

- Cusp forms: $S_k(\mathrm{Sl}_2(\mathbb{Z})) \subset \ker(\varphi_k)$.
- Derivatives: $q\frac{d}{dq}\mathcal{G}_{k-2} \subset \ker(\varphi_k)$.

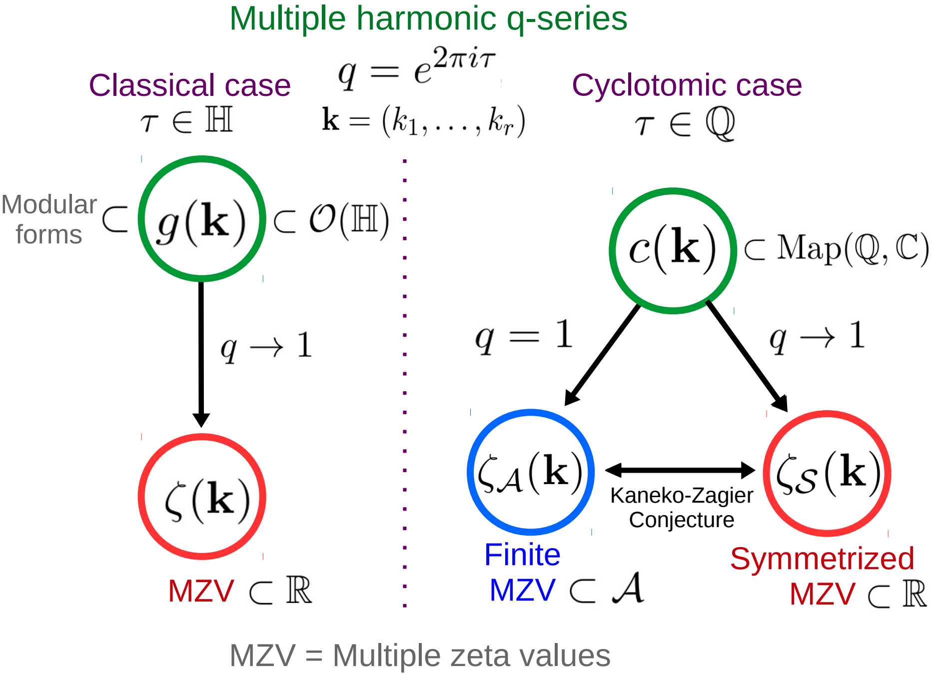
Example

One can prove that with $\alpha = -(2^6 \cdot 5 \cdot 691)^{-1}$

$$\alpha\Delta = 168\, g(5,7) + 150\, g(7,5) + 28\, g(9,3) + \frac{1}{1408}\, g(2) - \frac{83}{14400}\, g(4) + \frac{187}{6048}\, g(6) - \frac{7}{120}\, g(8) - \frac{5197}{691}\, g(12)\,.$$

Applying φ_{12} to this gives the Gangl-Kaneko-Zagier relation

$$168\zeta(5,7) + 150\zeta(7,5) + 28\zeta(9,3) = \frac{5197}{691}\zeta(12)\,.$$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Kaneko-Zagier Conjecture
Finite MZV $\subset \mathcal{A}$
Symmetrized MZV $\subset \mathbb{R}$
MZV = Multiple zeta values

From now on: $\tau \in \mathbb{Q}$, i.e. $q = e^{2\pi i\tau}$ is a root of unity.

$$\operatorname{ord}(q) = \min\{n \mid q^n = 1\}\,.$$

Definition

- For a root of unity q and any indexset $\mathbf{k} = (k_1, \dots, k_r)$ define

$$\begin{aligned}\mathrm{c}(\mathbf{k}; q) &= \mathrm{c}(k_1, \dots, k_r; q) \\ &= \sum_{\operatorname{ord}(q) > m_1 > \cdots > m_r > 0} \frac{P_{k_1}(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1-q^{m_r})^{k_r}}\end{aligned}$$

  and set $\mathrm{c}(\emptyset; q) = 1$.

- For the space spanned by all these functions we write

$$\mathcal{C} = \langle \mathrm{c}(\mathbf{k}) \rangle_{\mathbb{Q}} \subset \operatorname{Map}(\mathbb{Q}, \mathbb{C})$$

  and

$$\mathcal{C}_k = \langle \mathrm{c}(\mathbf{k}) \mid \operatorname{wt}(\mathbf{k}) \leq k \rangle_{\mathbb{Q}}\,.$$

The space $\mathcal{C}$ is also a $\mathbb{Q}$-algebra (with the same arguments as for $\mathcal{G}$).

Example

$$\mathrm{c}(2) \cdot \mathrm{c}(3) = \mathrm{c}(2,3) + \mathrm{c}(3,2) + \mathrm{c}(5) - \frac{1}{12}\,\mathrm{c}(3)\,.$$

We can prove relations between the c as elements in $\mathrm{Map}(\mathbb{Q}, \mathbb{C})$.

Example

$$\mathrm{c}(2,2) - 2\,\mathrm{c}(4) - \frac{1}{6}\,\mathrm{c}(2) = 0\,,$$
$$\mathrm{c}(4,1) - \mathrm{c}(1,4) + \mathrm{c}(3,2) = 0\,.$$

Notice that $\mathcal{C}$ is not graded by the weight.

Proposition

In depth 1 we have for $k \geq 1$

$$c(k;q) = (-1)^{k-1}\frac{B_k}{k!}(\mathrm{ord}(q)^k - 1)\,.$$

Now consider the case $\tau = \frac{1}{n}$, i.e. $q = e^{\frac{2\pi i}{n}} =: \xi_n$ and $\mathrm{ord}(q) = n$.

$$\lim_{n\to\infty}(1-\xi_n)n = \lim_{n\to\infty}\left(-\frac{2\pi i}{n} - \frac{1}{2}\left(\frac{2\pi i}{n}\right)^2 - \dots\right)n = -2\pi i\,.$$

Since $\zeta(k) = -\frac{B_k}{k!}\frac{(-2\pi i)^k}{2}$ for even k we obtain

$$\lim_{n\to\infty}(1-\xi_n)^k\, c(k;\xi_n) = \begin{cases} -\pi i & (k=1) \\ 2\zeta(k) & (k \geq 2,\ k \text{ is even}) \\ 0 & (k \geq 3,\ k \text{ is odd}) \end{cases}$$

In particular $\mathrm{Re}\left(\lim_{n\to\infty}(1-\xi_n)^k\; c(k;\xi_n)\right) = \zeta_{\mathcal{S}}(k)$.

③ Multiple harmonic q-series - cyclotomic case: Symmetrized MZV

$$q = \xi_n = e^{\frac{2\pi i}{n}}$$

Theorem (B.-Takeyama-Tasaka, 2017)

For any indexset $\mathbf{k} = (k_1, \dots, k_r)$ we have

$$L(\mathbf{k}) := \lim_{n\to\infty} (1-\xi_n)^{\mathrm{wt}(\mathbf{k})}\, \mathrm{c}(\mathbf{k};\xi_n)$$

$$= \sum_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta(k_a, k_{a-1}, \dots, k_1; -\frac{\pi i}{2}) \zeta(k_{a+1}, k_{a+2}, \dots, k_r; \frac{\pi i}{2}) .$$

In particular we have $L(\mathbf{k}) \in \mathcal{Z} + \pi i \mathcal{Z}$ and the following.

Corollary

For any indexset $\mathbf{k} = (k_1, \dots, k_r)$ we have

$$\mathrm{Re}(L(\mathbf{k})) \equiv \zeta_{\mathcal{S}}(\mathbf{k}) \mod \pi^2 \mathcal{Z} .$$

and $\mathrm{Re}(L(\mathbf{k})) = \zeta_{\mathcal{S}}(\mathbf{k})$ if $\mathbf{k} \neq (\dots, 1, 1, \dots)$.

The Theorem gives us a surjective linear map

$$\begin{aligned} \varphi_k^{\mathcal{S}} : \mathcal{C}_k &\longrightarrow \mathcal{Z}_k \\ \mathrm{c}(\mathbf{k}) &\longmapsto \mathrm{Re}\left(\lim_{n\to\infty} (1-\xi_n)^k \, \mathrm{c}(\mathbf{k};\xi_n) \right) , \end{aligned}$$

which satisfies $\varphi_k^{\mathcal{S}}(\mathrm{c}(\mathbf{k})) \equiv \zeta_{\mathcal{S}}(\mathbf{k}) \mod \pi^2\mathcal{Z}$.

Example

Relations between the c give relation between Symmetrized MZV (mod $\pi^2\mathcal{Z}$)

$$\mathrm{c}(4,1) - \mathrm{c}(1,4) + \mathrm{c}(3,2) = 0$$

$$\Big\downarrow \varphi_5^{\mathcal{S}}$$

$$\zeta_{\mathcal{S}}(4,1) - \zeta_{\mathcal{S}}(1,4) + \zeta_{\mathcal{S}}(3,2) = 0$$

For p prime let $\mathrm{I}_p = (1-\xi_p)$ be the ideal of $\mathbb{Z}[\xi_p]$ generated by $1-\xi_p$.

Lemma

Let p be a prime, then

- $(1-\xi_p)^k \, \mathrm{c}(\mathbf{k};\xi_p) \in \mathbb{Z}[\xi_p]$
- We have $\mathbb{Z}[\xi_p]/\,\mathrm{I}_p \cong \mathbb{F}_p$.

From this we get

Theorem (B.-Takeyama-Tasaka, 2017)

For any indexset $\mathbf{k} = (k_1, \dots, k_r)$ we have

$$\left((1-\xi_p)^k \, \mathrm{c}(\mathbf{k};\xi_p) \mod \mathrm{I}_p \right)_{p \text{ prime}} = \zeta_{\mathcal{A}}(\mathbf{k}) \,.$$

The Theorem gives us a surjective linear map

$$\begin{aligned}\varphi_k^{\mathcal{A}} : \mathcal{C}_k &\longrightarrow \mathcal{Z}_k^{\mathcal{A}} \\ \mathrm{c}(\mathbf{k}) &\longmapsto \Big((1-\xi_p)^k \, \mathrm{c}(\mathbf{k};\xi_p) \mod \mathrm{I}_p \Big)_{p \text{ prime}} ,\end{aligned}$$

which satisfies $\varphi_k^{\mathcal{A}}(\mathrm{c}(\mathbf{k})) = \zeta_{\mathcal{A}}(\mathbf{k})$.

Example

Relations between the c give relation between Finite MZV

$$\mathrm{c}(4,1) - \mathrm{c}(1,4) + \mathrm{c}(3,2) = 0$$

$$\Big\downarrow \varphi_5^{\mathcal{A}}$$

$$\zeta_{\mathcal{A}}(4,1) - \zeta_{\mathcal{A}}(1,4) + \zeta_{\mathcal{A}}(3,2) = 0$$

③ Kaneko-Zagier Conjecture - Revisited

The Kaneko-Zagier conjecture, i.e. φ_{KZ} in the diagram below is an isomorphism, implies the following commutative diagram

$$\begin{array}{ccc} & \mathcal{C}_k & \\ {\scriptstyle \varphi_k^{\mathcal{A}}} \swarrow & & \searrow {\scriptstyle \varphi_k^{\mathcal{S}}} \\ \mathcal{Z}_k^{\mathcal{A}} & \xrightarrow{\varphi_{KZ}} & \left(\mathcal{Z}/\zeta(2)\mathcal{Z}\right)_k \end{array}$$

So it is natural to conjecture the following.

Conjecture

We have $\ker \varphi_k^{\mathcal{A}} = \ker \varphi_k^{\mathcal{S}}$.

Question

How does the kernel of $\varphi_k^{\mathcal{A}}$ and $\varphi_k^{\mathcal{S}}$ look like?

Clearly we have $\mathcal{C}_{k-1} \subset \ker \varphi_k^{\mathcal{A}}$ and $\mathcal{C}_{k-1} \subset \ker \varphi_k^{\mathcal{S}}$, so we define

$$\operatorname{gr}_k \mathcal{C} = {}^{\mathcal{C}_k}\!/_{\mathcal{C}_{k-1}} .$$

By numerical experiments we get the following:

k	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{Q}} \operatorname{gr}_k \mathcal{C} \overset{?}{=}$	1	1	2	2	4	5	8	12	17	27	38	57
$\dim_{\mathbb{Q}} \mathcal{Z}_k^{\mathcal{A}} \overset{?}{=}$	0	0	1	0	1	1	1	2	2	3	4	5

Current work

Try to understand...

- ... the relations in $\mathcal{C}_k$.
- ... how to describe the kernel of $\varphi_k^{\mathcal{A}}$ and $\varphi_k^{\mathcal{S}}$.

Definition (Zagier)

A function $f : \mathbb{Q} \to \mathbb{C}$ is called a **quantum modular form** of weight k for $\Gamma \subset \mathrm{Sl}_2(\mathbb{Z})$, if the function $h_\gamma : \mathbb{Q}/\{\gamma^{-1}(\infty)\} \to \mathbb{C}$ defined by

$$h_\gamma(\tau) = f(\tau) - f_{|_k\gamma}(\tau)\,,$$

has some property of continuity or analyticity for every element $\gamma \in \Gamma$.

Here $f_{|_k\gamma}(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Define for $\gamma \in \mathrm{Sl}_2(\mathbb{Z})$ and an indexset $\mathbf{k}$

$$h_\gamma(\mathbf{k};\tau) = \mathrm{c}(\mathbf{k}; e^{2\pi i\tau}) - \mathrm{c}|_{\mathrm{wt}(\mathbf{k})}\gamma(\mathbf{k}; e^{2\pi i\tau})$$

and write $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proposition

For $k \geq 1$ we have

$$h_S(k;\tau) = (-1)^{k-1}\frac{B_k}{k!} \cdot \left(\frac{1}{\tau^k} - 1\right),$$

i.e. $\mathrm{c}(k)$ is a quantum modular form of weight k.

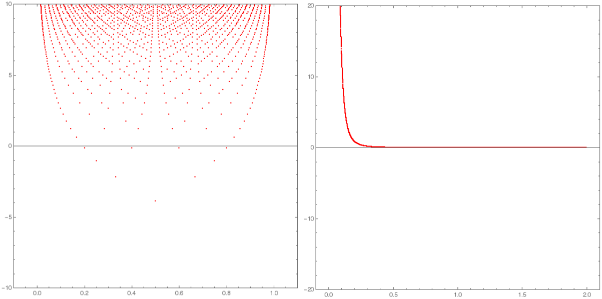
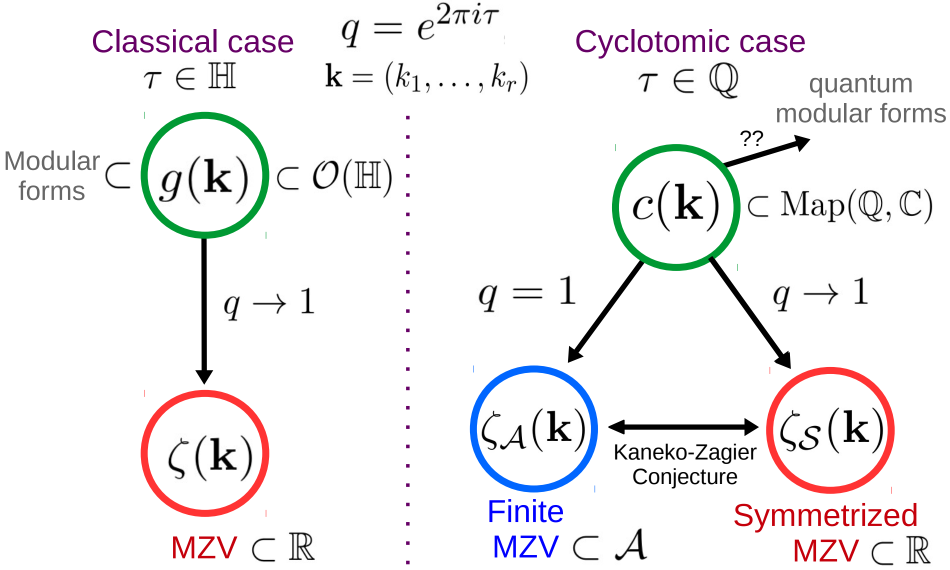


Figure: The graphs of $\mathrm{c}(4;\tau)$ and $h_S(4;\tau) = \mathrm{c}(4;\tau) - \frac{1}{\tau^4}\,\mathrm{c}\left(4; -\frac{1}{\tau}\right)$

Question

- Are there other examples in higher depths, which give quantum modular forms?
- Does this have any application/meaning for finite & symmetrized MZV?



Thank you very much for your attention!

Slides are available on my homepage: **www.henrikbachmann.com**