Combinatorial multiple Eisenstein series and polynomial functions on partitions

Overview: Kyushu University 6th - 8th April 2022

(1) Multiple zeta values, Multiple Eisenstein series and q-analogues of multiple zeta values

- 1.1 MZV
- 1.2 Algebraic setup I & Regularization
- 1.3 MES & qMZV
- 1.4 MES as maps and Stuffle & Shuffle reg. MES

(2) Combinatorial MES

- 2.1 Motivation & Overview
- 2.2 Algebraic setup II (Bimoulds)
- 2.3 The bimoulds $\mathcal{B}$, $\mathcal{L}_m$ and $\mathcal{G}^*$
- 2.4 The bimould $\mathcal{G}$ and CMES
- 2.5 Proof sketch

(3) Pol. fct on partitions & bi-MZV

- 3.1 Introduction & Fct. on partitions
- 3.2 Pol. fct and their degree limit.
- 3.3 Algebraic Setup III
- 3.4 Bi-MZV & Application

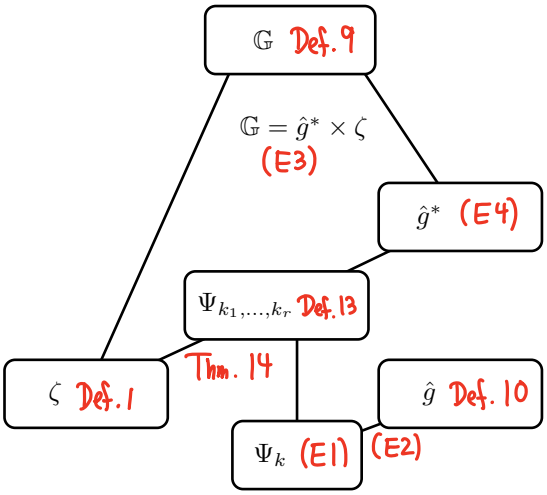
Combinatorial multiple Eisenstein series & Polynomial functions on partitions
Lecture series at the Multiple Zeta Research Center, Kyushu University, 6th April - 8th April 2022
Henrik Bachmann, Nagoya University

An overview, this handout, the handwritten lectures notes, and references can be found on

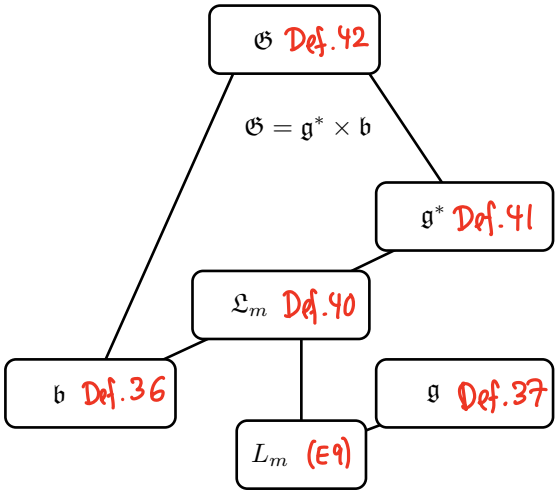
`https://www.henrikbachmann.com/cmes.html`.

If you have any questions, you can contact me anytime via email: henrik.bachmann@math.nagoya-u.ac.jp

Part 1: Multiple zeta values, Multiple Eisenstein series and q-analogues (6th - 7th April)



Part 2: Combinatorial multiple Eisenstein series (7th - 8th April)



Part 3: Polynomial functions on partitions and bi-multiple zeta values (8th April)

$$\begin{array}{ccccc} \mathbb{Q}^{\mathscr{P}} & \xrightarrow{\langle\ \rangle_q} & \mathbb{Q}[\![q]\!] & \xrightarrow{\text{“}\lim_{q\to 1}\text{”}} & \mathbb{R} \\ \cup & & \cup & & \cup \\ \mathbb{P} & \longrightarrow & \mathcal{Z}_q & \longrightarrow & \mathcal{Z} \\ \cup & & \cup & & \cup \\ \Lambda^* \subset \mathbb{M} & \longrightarrow & \widetilde{M} & \longrightarrow & \mathbb{Q}[\zeta(2)]. \end{array}$$

References

[B1] H. Bachmann: *The algebra of bi-brackets and regularized multiple Eisenstein series*, J. Number Theory, **200** (2019), 260–294.

[B2] H. Bachmann: *Multiple Eisenstein series and q-analogues of multiple zeta values*, Periods in Quantum Field Theory and Arithmetic, Springer Proceedings in Mathematics & Statistics **314** (2020), 173–235.

[B3] H. Bachmann: *q-analogues of multiple zeta values and the formal double Eisenstein space*, Waseda Number Theory Conference Proceedings 2021, arXiv:2108.08634.

[B4] H. Bachmann: *Multiple zeta values and modular forms*, Lecture notes (Ver. 5.4), Nagoya University 2020. Available online at `https://www.henrikbachmann.com/mzv2020.html`.

[BB] H. Bachmann, A. Burmester: *Combinatorial multiple Eisenstein series*, preprint, arXiv:2203.17074.

[BI] H. Bachmann, J.W.M. v. Ittersum: *Partitions, Multiple Zeta Values and the q-bracket*, preprint, arXiv:2203.09165.

[BIM] H. Bachmann, J.W.M. v. Ittersum, N. Matthes: *Formal multiple Eisenstein series*, in preparation.

[BK] H. Bachmann, U. Kühn: *The algebra of generating functions for multiple divisor sums and applications to multiple zeta values*, Ramanujan J. **40** (2016), 605–648.

[BK2] H. Bachmann, U. Kühn: *A dimension conjecture for q-analogues of multiple zeta values*, Theory and Arithmetic, Springer Proceedings in Mathematics & Statistics **314** (2020), 237–258.

[BKM] H. Bachmann, U. Kühn, N. Matthes: *Realizations of the formal double Eisenstein space*, preprint, arXiv:2109.04267.

[BT] H. Bachmann, K. Tasaka: *The double shuffle relations for multiple Eisenstein series*, Nagoya Math. Journal **230** (2017), 1–33.

[BO] S. Bloch, A. Okounkov: *The character of the infinite wedge representation*, Adv. Math. **149**:1 (2000), 1–60

[Bo] O. Bouillot: *The algebra of multi tangent functions*, J. Algebra **410** (2014), 148–238.

[Bo2] O. Bouillot: *Mould calculus: from primary to secondary mould symmetries* , IRMA Lectures in Mathematics and Theoretical Physics **31**, "Algebraic Combinatorics, Resurgence, Moulds and Applications (CARMA) Vol. 2", 19–84.

[Bri] B. Brindle: *A unified approach to qMZVs*, preprint, arXiv:2111.00051.

[E] J. Ecalle: *ARI/GARI, la dimorphie et l'arithmétique des multizêtas : un premier bilan*, Journal de Théorie des Nombres de Bordeaux, Tome **15** (2003) no. 2, 411–478.

[GKZ] H. Gangl, M. Kaneko, D. Zagier: *Double zeta values and modular forms*, in "Automorphic forms and zeta functions" World Sci. Publ., Hackensack, NJ (2006), 71–106.

[H] M. E. Hoffman: *Quasi-shuffle products*, J. Algebraic Combin. **11** (2000), 49–68.

[HI] M. E. Hoffman, K. Ihara: *Quasi-shuffle products revisited*, J. Algebra **481** (2017), 293–326.

[IKZ] K. Ihara, M. Kaneko, D. Zagier: *Derivation and double shuffle relations for multiple zeta values*, Compositio Math. **142** (2006), 307–338.

[I] J.W.M. v. Ittersum: *A symmetric Bloch–Okounkov Theorem*, Res. Math. Sci. **8**, 19 (2021).

[Za] D. Zagier: *Partitions, quasimodular forms, and the Bloch–Okounkov theorem*, Ramanujan J. **41** (2016), no. 1-3, 345–368.

① Multiple zeta values (MZV), Multiple Eisenstein series (MES) and q-analogues of MZV (qMZV)

1.1 MZV

Definition 1: For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the MZV

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \ldots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$

weight: $k_1 + \ldots + k_r$ $\quad (r=0: \ \zeta(\emptyset) := 1)$

depth: r

$$Z = \langle \zeta(k_1, \ldots, k_r) \mid r \geq 0, k_1 \geq 2, k_2, \ldots, k_r \geq 1 \rangle_{\mathbb{Q}}$$
$$Z_k = \langle \zeta(k_1, \ldots, k_r) \mid r \geq 0, k_1 \geq 2, k_2, \ldots, k_r \geq 1, k_1 + \ldots + k_r = k \rangle_{\mathbb{Q}}$$

Z is a $\mathbb{Q}$-algebra, e.g.

$(k_1, k_2 \geq 2)$ (harmonic/stuffle product)

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2)$$
$$= \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j)$$

Using iterated integral expression or partial fraction decomp.

(shuffle product)

- The relations obtained from writing the product of MZV in two different ways are called (finite) double shuffle relations (FDSH)

- MZV satisfy a lot of relations.

Conjecture (Zagier) $\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k X^k = \frac{1}{1-X^2-X^3}$

k	0	1	2	3	4	5	6	7	8
## indices	1	0	1	2	4	8	16	32	64
$\dim_{\mathbb{Q}} \mathcal{Z}_k \overset{?}{=}$	1	0	1	(1)	1	2	2	3	1

- Not all of them follow from FDSH, e.g. $\boxed{\zeta(3)=\zeta(2,1)}$
  ⇝ regularization & extended double shuffle rel.

- There exist also various explicit evaluations, e.g.

(Euler) k even: $\zeta(k) = -\frac{B_k}{2k!}(2\pi i)^k$

$\zeta(\overbrace{2,\dots,2}^{n}) =: \zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}$ $\qquad \zeta(\{3,1\}^n) = \frac{2\pi^{4n}}{(4n+2)!}$

- We will describe these products by using the notion of quasi-shuffle products.

1.2 Algebraic setup (Quasi-shuffle products) & Regularization

L: countable set (letters)

monic monomial in $\mathbb{Q}\langle L\rangle$: word e.g. $L=\{a,b,n\}$

$\mathbf{1}$: empty word

$\frac{1}{-24} - \frac{1}{2}\,abba + 3\,banana \in \mathbb{Q}\langle L\rangle$

$= -\frac{1}{24}\cdot\mathbf{1}$ word

$\diamond$: commutative & associative product on $\mathbb{Q}L$.

Definition 2 We define the quasi-shuffle product $*_\diamond$ as the $\mathbb{Q}$-bilinear product, which satisfies $\mathbf{1} *_\diamond w = w *_\diamond \mathbf{1}$ for any word $w \in \mathbb{Q}\langle L\rangle$ and

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a\diamond b)(w *_\diamond v)$$

for any letters $a,b \in L$ and words $w \in \mathbb{Q}\langle L\rangle$.

Theorem 3 (Hoffman)

- $(\mathbb{Q}\langle L\rangle, *_\diamond)$ is a commutative $\mathbb{Q}$-algebra
- This algebra can be equipped with the structure of a Hopf algebra.

deconcatination coproduct:

$$\Delta(w) = \sum_{w \text{ word}} \sum_{uv=w} u \otimes v$$

e.g. $\Delta(aba) = \mathbf{1}\otimes aba + a\otimes ba + ab\otimes a + aba\otimes\mathbf{1}$.

We are interested in the following cases:

(i) $L_z = \{z_k \mid k \geq 1\}$, $z_{k_1} \circ z_{k_2} = z_{k_1+k_2} \quad \forall k_1, k_2 \geq 1$

$\mathfrak{h}' := \mathbb{Q}\langle L_z \rangle$, $* := *_{\circ}$ stuffle product

$\mathfrak{h}'_* := (\mathfrak{h}', *)$

$\mathfrak{h}^0 = \langle z_{k_1} \cdots z_{k_r} \mid r \geq 0, k_1 \geq 2 \rangle_{\mathbb{Q}} \subset \mathfrak{h}^0$

$\mathfrak{h}^0_* := (\mathfrak{h}^0, *)$ is a subalgebra of $\mathfrak{h}'_*$.

Now define the $\mathbb{Q}$-linear map

$$\zeta : \mathfrak{h}^0 \longrightarrow \mathbb{Z}$$
$$z_{k_1} \cdots z_{k_r} \longmapsto \zeta(k_1, \ldots, k_r).$$

Proposition 4: $\zeta : \mathfrak{h}^0_* \to \mathbb{Z}$ is an alg. homomorphism

(ii) $L_{xy} = \{x, y\}$, $z_a \circ z_b = 0 \quad \forall a, b \in L_{xy}$.

$\mathfrak{h} := \mathbb{Q}\langle L_{xy} \rangle$, $ш := *_{\circ}$ shuffle product

$\mathfrak{h}_ш := (\mathfrak{h}, ш)$

Via $z_k \hookrightarrow \overbrace{x \cdots x}^{k-1} y$ we can view $\mathfrak{h}^0, \mathfrak{h}^1$ as subspaces of $\mathfrak{h}$, i.e
$$\mathfrak{h}^0 \cong \mathbb{Q} + x\mathfrak{h}y$$
$$\mathfrak{h}^1 \cong \mathbb{Q} + \mathfrak{h}y.$$

Both are closed under ш and we have the following inclusions of $\mathbb{Q}$-algebras
$$\underset{(\mathfrak{h}^0, ш)}{\mathfrak{h}^0_ш} \subset \underset{(\mathfrak{h}^1, ш)}{\mathfrak{h}^1_ш} \subset \mathfrak{h}_ш.$$

Proposition 5 $\zeta : \mathfrak{h}^0_ш \to \mathbb{Z}$ is an alg. homomorphism

(Proof uses iterated integrals, e.g.
$$\zeta(2,3) = \zeta(z_2 z_3) = \zeta(xy\,xxy) = \underset{x}{\int_0^1 \frac{dt_1}{t_1}} \underset{y}{\int_0^{t_1} \frac{dt_2}{1-t_2}} \underset{x}{\int_0^{t_2} \frac{dt_3}{t_3}} \underset{x}{\int_0^{t_3} \frac{dt_4}{t_4}} \underset{y}{\int_0^{t_4} \frac{dt_5}{1-t_5}}$$
)

__FDSH__: For all $w, v \in \mathfrak{h}^0$ we have
$$\zeta(w * v - w ш v) = 0$$

We now want to extend the map ζ to $\mathfrak{h}^1$. This can be done such that this either gives an alg. hom from $\mathfrak{h}^1_*$ (denoted ζ^*) or $\mathfrak{h}^1_ш$ ($\zeta^ш$)

Example 6 $z_2 * z_1 = z_2 z_1 + z_1 z_2 + z_3$

$$\Rightarrow \quad z_1 z_2 = z_2 * z_1 - z_2 z_1 - z_3$$

$$\rightsquigarrow \varphi^*(1,2) = \varphi^*(z_1 z_2) = \varphi^*(z_2)\varphi^*(z_1) - \varphi^*(z_2 z_1) - \varphi^*(z_3)$$

$$= \varphi(2)\underbrace{\varphi^*(z_1)}_{\text{only choice}} - \varphi(2,1) - \varphi(3).$$

$$z_2 \sqcup\!\sqcup z_1 = xy \sqcup\!\sqcup y = 2xyy + yxy$$

$$\Rightarrow \quad yxy = xy \sqcup\!\sqcup y - 2xyy$$

$$z_1 z_2 = z_2 \sqcup\!\sqcup z_1 - 2 z_2 z_1$$

$$\rightsquigarrow \varphi^{\sqcup\!\sqcup}(1,2) = \varphi(2)\varphi^{\sqcup\!\sqcup}(z_1) - 2\varphi(2,1)$$

Proposition 7

We have $\mathfrak{h}'_* = \mathfrak{h}^0_*[z_1]$ and $\mathfrak{h}'_{\sqcup\!\sqcup} = \mathfrak{h}^0_{\sqcup\!\sqcup}[\underbrace{y}_{\| \; z_1}]$.

So any $w = z_{k_1} \cdots z_{k_r} \in \mathfrak{h}'$ can be uniquely written as

$$w = \sum_{j=0}^{r} w_j^* * z_1^{*j} = \sum_{j=0}^{r} w_j^{\sqcup\!\sqcup} \sqcup\!\sqcup y^{\sqcup\!\sqcup j}.$$

with $w_j^*, w_j^{\sqcup\!\sqcup} \in \mathfrak{h}^0$. (In general $w_j^* \neq w_j^{\sqcup\!\sqcup}$)

With this we define for $\bullet \in \{*, \mathrm{III}\}$

$$\zeta_T^\bullet : \mathfrak{h}' \longrightarrow \mathbb{Z}[T]$$
$$w \longmapsto \sum_{j=0}^{r} \zeta^\bullet(w_j) T^j .$$

(We set $\zeta^\bullet(k_1, \dots, k_r) := \zeta^\bullet(z_{k_1} \cdots z_{k_r})$)

By construction $\zeta^\bullet$ is an alg. hom. from $\mathfrak{h}'$ to $\mathbb{Z}[T]$.

Example 6 $\Rightarrow$ $\zeta_T^*(1,2) = \zeta(2)T - \zeta(2,1) - \zeta(3)$

$$\zeta_T^{\mathrm{III}}(1,2) = \zeta(2)T - 2\,\zeta(2,1)$$

Assuming we already know that $\zeta(2,1) = \zeta(3)$, we see $\zeta_T^*(1,2) = \zeta_T^{\mathrm{III}}(1,2)$. But this is not true in general.

Theorem 8 (Ihara-Kaneko-Zagier)

Define the $\mathbb{R}$-linear map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$

by $$\rho\left(e^{Tu}\right) = \exp\left(Tu + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n) u^n\right).$$

($\rho(e^{Tu}) = \rho(1) + \rho(T)u + \frac{1}{2}\rho(T^2)u^2 + \dots$ and $\exp(\dots) = 1 + Tu + \frac{1}{2}\left(T^2 + \zeta(2)\right)u^2 + \dots$)

Then $$\zeta^{\mathrm{III}} = \rho \circ \zeta^* .$$

□

Since $\varphi(1)=1$, $\varphi(T)=T$ this implies $\zeta^*_T(1,2)=\zeta^{\text{III}}_T(1,2)$.

The relations obtained from this theorem are called extended double shuffle relations. (EDSH)

If we multiply $\bar{\varphi}^{-1}(\zeta^{\text{III}}(w))=\zeta^*(w)$ $(w\in\mathfrak{h}')$ with $\zeta(v)\in\mathbb{R}$ $(v\in\mathfrak{h}^0)$ we get by the linearity of $\bar{\varphi}^{-1}$:

$$\begin{array}{ccc}
\zeta(v)\cdot\bar{\varphi}^{-1}(\zeta^{\text{III}}(w)) & = & \zeta(v)\,\zeta^*(w) \\
\| & & \| \\
\bar{\varphi}^{-1}(\zeta(v)\cdot\zeta^{\text{III}}(w)) & & \zeta^*(v*w) \\
\| & & \| \\
\bar{\varphi}^{-1}(\zeta^{\text{III}}(v\,\text{III}\,w)) & = & \zeta^*(v\,\text{III}\,w)
\end{array}$$

$\rightsquigarrow$ One can show: The EDSH are equivalent to

$$\zeta^*(v\,\text{III}\,w - v*w)=0$$

for $v\in\mathfrak{h}^0$ and $w\in\mathfrak{h}'$.

(Conj: $w\in\{z_1,z_2,z_3,z_2z_1\}$ is enough)

1.3 MES and qMZV

The MZV in depth $r=1$ appear as the constant term of the Eisenstein series $(k \geq 4$ even$)$

$$G_k(\tau) = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2 \setminus (0,0)} \frac{1}{(m\tau+n)^k} \overset{\text{later}}{=} \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, $q = e^{2\pi i \tau}$ and $\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}$.

To define their multiple versions we first define an order $\succ$ on $\mathbb{Z}\tau + \mathbb{Z}$ as follows

$$m_1\tau + n_1 \succ m_2\tau + n_2 \overset{\text{def}}{:\Longleftrightarrow} m_1 > m_2 \text{ or } m_1 = m_2 \wedge n_1 > n_2$$

Definition 9 ([GKZ]) For $k_1 \geq 3$ ($k_1 \geq 2$), $k_2, \dots, k_r \geq 2$ we define the multiple Eisenstein series by

$$G_{k_1,\dots,k_r}(\tau) = \lim_{M\to\infty} \lim_{N\to\infty} \sum_{\substack{\lambda_1 > \dots > \lambda_r > 0 \\ \lambda_i \in \frac{1}{M}\mathbb{Z}\tau + \mathbb{Z}_N}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}.$$

r: depth

$k_1 + \dots + k_r$: weight

$\mathbb{Z}_M = \{ m \in \mathbb{Z} \mid |m| < M \}$

- These also satisfy the stuffle product formula (later more)

- Further notice that $G_{k_1,\dots,k_r}(\tau+1) = G_{k_1,\dots,k_r}(\tau)$ and

$$\lim_{\tau \to i\infty} G_{k_1,\dots,k_r}(\tau) = \zeta(k_1,\dots,k_r).$$

$$\Rightarrow \exists\, a_n \in \mathbb{C} \quad G_{k_1,\dots,k_r}(\tau) = \zeta(k_1,\dots,k_r) + \sum_{n=1}^{\infty} a_n q^n$$

where $q = e^{2\pi i \tau}$.

We explain the construction of a_n in the following.

$r = 1$:

$k \geq 3$ $\quad G_k(\tau) = \sum_{m\tau+n>0} \frac{1}{(m\tau+n)^k} =$ $\zeta(k) + \sum_{m>0} \underbrace{\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}}_{\Psi_k(m\tau)}$

$m = 0$, $n > 0$

Here we define the monotangent function $\Psi_k(\tau)$ for $k \geq 2$ by

$$\Psi_k(\tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)^k} = (-2\pi i)^k \sum_{n>0} \frac{n^{k-1}}{(k-1)!} q^n \quad (\in I)$$

(Lipschitz formula)

$$\Rightarrow \quad G_k(\tau) = \zeta(k) + \underbrace{\sum_{m>0} \Psi_k(m\tau)}_{(-2\pi i)^k \sum_{\substack{m>0 \\ n>0}} \frac{n^{k-1}}{(k-1)!} q^{md}} = \zeta(k) + \underbrace{(-2\pi i)^k g(k)}_{\hat{g}(k)}.$$

$$g(k) :=$$

Multiple versions of g will be needed later.

Definition 10 For $k_1, \ldots, k_r \geq 1$ we define

$$g(k_1, \ldots, k_r) := \sum_{\substack{m_1 > \ldots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \frac{n_1^{k_1-1}}{(k_1-1)!} \cdots \frac{n_r^{k_r-1}}{(k_r-1)!} q^{m_1 n_1 + \ldots + m_r n_r} \quad (\in \mathbb{Q}[[q]])$$

"$g(k_1, \ldots, k_r; q)$"

and write

$$\hat{g}(k_1,\dots,k_r) := (-2\pi i)^{k_1+\dots+k_r} g(k_1,\dots,k_r). \qquad \square$$

The g can be seen as (modified) q-analogues of MZV:

Proposition 11 For $k_1 \geq 2$ we have

$$\lim_{q\to 1} (1-q)^{k_1+\dots+k_r} g(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r). \qquad \square$$

Proof: We have

$$\sum_{n>0} \frac{n^{k-1}}{(k-1)!} q^{mn} = \frac{P_k(q^m)}{(1-q^m)^k},$$

where $P_k(x) \in \mathbb{Q}[x]$ are (normalized) Eulerian polynomials satisfying $P_k(1) = 1$.

$$\Rightarrow (1-q)^{k_1+\dots+k_r} g(k_1,\dots,k_r) = \sum_{m_1>\dots>m_r>0} \frac{P_{k_1}(q^{m_1})}{[m_1]_q^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{[m_r]_q^{k_r}}$$

with the q-integer $[m]_q = \frac{1-q^m}{1-q} = 1+q+\dots+q^{m-1} \xrightarrow[q\to 1]{} m.$ $\square$

As a generalization of $G_k(\tau) = \zeta(k) + \hat{g}(k)$ we have the following.

Theorem 12 ([GKZ] $r=1,2$, B. $r \geq 2$)

For $k_1 \geq 3$, $k_2, \dots, k_r \geq 2$ there exist explicit $\alpha_{\ell_1,\dots,\ell_j}^{k_1,\dots,k_r} \in \mathbb{Q}$ (∈ℚ) s.th. ($k_1 \geq 2$)

$$G_{k_1,\dots,k_r}(\tau) = \zeta(k_1,\dots,k_r) + \sum_{0<j<r} \alpha_{\ell_1,\dots,\ell_j}^{k_1,\dots,k_r} \zeta(\ell_1,\dots,\ell_j)\hat{g}(\ell_{j+1},\dots,\ell_r) + \hat{g}(k_1,\dots,k_r).$$
$$\ell_1+\dots+\ell_r = k_1+\dots+k_r$$

We will sketch the proof of this in the following.

First notice that for $k_1,\dots,k_r \geq 2$ we have by (E1)

$$\hat{g}(k_1,\dots,k_r) = \sum_{m_1 > \dots > m_r > 0} \psi_{k_1}(m_1\tau) \cdots \psi_{k_r}(m_r\tau). \quad (E2)$$

G will be written by similar sums, with multiple versions of ψ_k:

Definition 13 (Ecalle / Bouillot) For $k_1, \dots, k_r \geq 2$ we define the multitangent functions by

$$\psi_{k_1,\dots,k_r}(\tau) = \sum_{\substack{n_1 > \dots > n_r \\ n_i \in \mathbb{Z}}} \frac{1}{(\tau+n_1)^{k_1} \cdots (\tau+n_r)^{k_r}}.$$

$(\in G(\mathbb{C}\setminus\mathbb{Z}))$

How do they appear in the G?

$r=2$:

$$G_{k_1,k_2}(\tau) = \sum_{m_1\tau+n_1 > m_2\tau+n_2 > 0} \frac{1}{(m_1\tau+n_1)^{k_1}(m_2\tau+n_2)^{k_2}}$$

$m_1 > m_2$ or $m_1 = m_2 \wedge n_1 > n_2$; $m_2 > 0$ or $m_2 = 0 \wedge n_2 > 0$

$$= \textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4}$$

① $m_1 = m_2 = 0$, $n_1 > n_2 > 0$ $\rightsquigarrow \zeta(k_1,k_2)$

② $m_1 > m_2 = 0$, $n_2 > 0$

③ $m_1 = m_2 > 0$, $n_1 > n_2$

④ $m_1 > m_2 > 0$

$$② = \sum_{m_1>0} \underbrace{\sum_{n_1\in\mathbb{Z}} \frac{1}{(m_1\tau+n)^{\kappa_1}}}_{\psi_{\kappa_1}(m_1\tau)} \cdot \mathcal{C}(\kappa_2) = \hat{g}(\kappa_1)\,\mathcal{C}(\kappa_2).$$

$$③ = \sum_{m=m_1=m_2>0} \underbrace{\sum_{n_1>n_2} \frac{1}{(m\tau+n_1)^{\kappa_1}(m\tau+n_2)^{\kappa_2}}}_{\psi_{\kappa_1,\kappa_2}(m\tau)} = \sum_{m>0} \psi_{\kappa_1,\kappa_2}(m\tau).$$

$$④ = \sum_{m_1>m_2>0} \underbrace{\sum_{n_1,n_2\in\mathbb{Z}} \frac{1}{(m_1\tau+n_1)^{\kappa_1}(m_2\tau+n_2)^{\kappa_2}}}_{\psi_{\kappa_1}(m_1\tau)\,\psi_{\kappa_2}(m_2\tau)} = \hat{g}(\kappa_1,\kappa_2).$$

$$G_{\kappa_1,\kappa_2}(\tau) = \mathcal{C}(\kappa_1,\kappa_2) + g(\kappa_1)\mathcal{C}(\kappa_2) + \overbrace{\sum_{m>0} \psi_{\kappa_1,\kappa_2}(m\tau) + \hat{g}(\kappa_1,\kappa_2)}^{g^*(\kappa_1,\kappa_2)}.$$

To write this in terms of $\mathcal{C}$ and g we need the following.

Theorem 14 (Bouillot)

For $k_1, \dots, k_r \geq 2$ we have $(k = k_1 + \dots + k_r)$

$$\psi_{k_1,\dots,k_r}(\tau) = \sum_{j=2}^{k} \beta_j^{k_1,\dots,k_r} \psi_j(\tau)$$

with $\beta_j^{k_1,\dots,k_r} \in \mathbb{Z}_{k-j}$

"$\psi_{k_1 \dots k_r} = \mathcal{U} \times \psi \times \overleftarrow{\mathcal{U}}$"

$$\left(\psi_{k_1,\dots,k_r}(\tau) = \sum_{\substack{1 \leq j \leq r \\ \ell_1 + \dots + \ell_r = k_1 + \dots + k_r \\ k}} (-1)^{\ell_1 + \dots + \ell_{j-1} + k_j + k} \prod_{\substack{i=1 \\ i \neq j}}^{r} \binom{\ell_i - 1}{k_i - 1} \mathcal{U}(\ell_1, \dots, \ell_{j-1}) \psi_{\ell_j} \mathcal{U}(\ell_r, \ell_{r-1}, \dots, \ell_{j+1}) \right)$$

The proof uses partial fraction decomposition and a relation among MZV to show that the sum starts at $j=2$. $(\beta_1^* = 0)$

Example 15

$$\psi_{3,2}(\tau) = \sum_{n_1 > n_2} \frac{1}{(\tau + n_1)^3 (\tau + n_2)^2}$$

$$= \sum_{n_1 > n_2} \left(\frac{1}{(n_1 - n_2)^2 (\tau + n_1)^3} + \frac{2}{(n_1 - n_2)^3 (\tau + n_1)^2} + \frac{3}{(n_1 - n_2)^4 (\tau + n_1)} \right)$$

$$= \sum_{n_1 > n_2} \left(\frac{1}{(n_1 - n_2)^3 (\tau + n_2)^2} - \frac{3}{(n_1 - n_2)^4 (\tau + n_2)} \right)$$

$$= 3\,\zeta(3)\,\Psi_2(\tau) + \zeta(2)\,\Psi_3(\tau).$$

$$\Rightarrow \sum_{m>0} \Psi_{3,2}(m\tau) = 3\,\zeta(3)\,\hat{g}(2) + \zeta(2)\,\hat{g}(3).$$

$$\alpha^{3,2}_{3,2,1} \qquad \alpha^{3,2}_{2,3,1}$$

$$\Rightarrow G_{3,2}(\tau) = \zeta(3,2) + \tilde{3}\,\zeta(3)\,\hat{g}(2) + \tilde{2}\,\zeta(2)\,\hat{g}(3) + \hat{g}(3,2)$$

For general depth we have

$$m_1 \geq \dots \geq m_j > \quad m_{j+1} = \dots = m_r = 0$$

$$G_{k_1,\dots,k_r}(\tau) = \sum_{j=0}^{r} \hat{g}^*(k_1,\dots,k_j)\,\zeta(k_{j+1},\dots,k_r), \qquad (E3)$$

where

$$\hat{g}^*(k_1,\dots,k_r) := \sum_{\substack{1\le j\le r \\ 0=r_0<r_1<\cdots<r_{j-1}<r_j=r \\ m_1>\cdots>m_j>0}} \prod_{i=1}^{j} \Psi_{k_{r_{i-1}+1},\dots,k_{r_i}}(m_i\tau). \qquad (E4)$$

j = # of ">" in "$m_1 \geq \dots \geq m_j >$"

e.g. $j=1$: $0 = r_0 < r_1 = r$ $\quad \Psi_{k_1,\dots,k_r}(m_1\tau)$

Example 16

$$\widehat{g}^*(k_1,k_2,k_3) = \sum_{m>0} \psi_{k_1,k_2,k_3}(m\tau) \qquad j=1$$

$$+ \sum_{m_1>m_2>0} \left(\psi_{k_1}(m_1\tau)\psi_{k_2,k_3}(m_2\tau) + \psi_{k_1,k_2}(m_1\tau)\psi_{k_3}(m_2\tau) \right) \qquad j=2$$

$$0 \underset{r_0}{<} 1 \underset{r_1}{<} 3 \underset{r_2}{} \qquad\qquad 0 \underset{r_0}{<} 2 \underset{r_1}{<} 3 \underset{r_2}{}$$

$$+ \sum_{m_1>m_2>m_3>0} \psi_{k_1}(m_1\tau)\psi_{k_2}(m_2\tau)\psi_{k_3}(m_3\tau) \qquad j=3$$

$$\widehat{g}(k_1,k_2,k_3)$$

Fourier expansion of $G_{k_1,\dots,k_r}$ (Theorem 12)

G Def. 9

"$G = \hat{g}^* \times \mathcal{U}$" (E3)

(E4) $g^* = \sum \psi_{*,\dots,*}\psi_{x,\dots,*}$

Def 13 "$\psi_{k_1,\dots,k_r} = \mathcal{U} \times \psi \times \overleftarrow{\mathcal{U}}$"

(E2) $g = \sum \psi_{k_1} \cdots \psi_{k_r}$

Def 1 $\mathcal{U}$

Thm. 14

(E1) ψ_k

1.4 MES as maps and G^* & G^{III} $\left(\mathfrak{h}_*^G \subset \mathfrak{h}_*^0 \subset \mathfrak{h}_*^1\right)$

Define the Hopf subalgebra $\mathfrak{h}_*^G = \mathbb{Q}\langle z_k \mid k \geq 2\rangle \subset \mathfrak{h}_*^1$.

Then the MES can be viewed as a $\mathbb{Q}$-linear map

$$G: \mathfrak{h}_*^G \longrightarrow \mathbb{C}[[q]] \quad (\text{or } G(\mathbb{H}))$$

$$z_{k_1} \cdots z_{k_r} \longmapsto G_{k_1,\ldots,k_r}$$

Definition as iterated sum ("Convergence problems") $\Longrightarrow$ G is an alg. hom.

Can we obtain this from the properties of $\mathfrak{a}$, $\hat{g}^*$ or Ψ?

We can also view $\mathfrak{a}, g, \hat{g}^*, \Psi$ as linear maps $\mathfrak{a}: \mathfrak{h}^G \longrightarrow \mathbb{Z}$,

$$\hat{g}, \hat{g}^*: \mathfrak{h}^G \longrightarrow \mathbb{Q}[2\pi i][[s]],$$
$$\Psi: \mathfrak{h}^G \longrightarrow G(\mathbb{H}).$$

By (E3) G is the convolution product of $\mathfrak{a}$ and $\hat{g}^*$

$$G = \hat{g}^* * \mathfrak{a} := m \circ (\hat{g}^* \otimes \mathfrak{a}) \circ \Delta,$$

where $m: \mathbb{C}[[q]] \otimes \mathbb{C}[[s]] \longrightarrow \mathbb{C}[[q]]$ is the usual multiplication.

Example 17

$$\mathcal{G}(z_3 z_2) = \left(m\circ(\hat{g}^* \otimes \mathcal{U})\circ\Delta\right)(z_3 z_2)$$

$$= m\left((\hat{g}^* \otimes \mathcal{U})(1\otimes z_3 z_2 + z_3 \otimes z_2 + z_3 z_2 \otimes 1)\right)$$

$$= \mathcal{U}(z_3 z_2) + \hat{g}^*(z_3)\,\mathcal{U}(z_2) + \hat{g}^*(z_3 z_2).$$

General: If $f, g \in \underset{\text{alg. hom}}{\text{Hom}}(\underset{\text{Hopf alg.}}{H}, \underset{\text{some alg.}}{A})$ then $f \star g \in \text{Hom}(H, A)$.

- $\mathcal{U}$ and Ψ are alg. hom. (def as iterated sum).
- What about $\hat{g}^*$? ← Will follow from the following: (Compare to def of $\hat{g}^*$) (EY)

Lemma 18

Let $f_m : H' \longrightarrow A$ be a family of alg. hom. for $m \geq 1$. (H': Hopf subalgebra of $\mathfrak{h}'$; A: $\mathbb{Q}$-alg.) Then for $M > 1$ the $\mathbb{Q}$-linear map F_M defined on a word $w \in H$ by ($F_M(\mathbb{1}) = 1$)

$$F_M(w) = \sum_{\substack{(\text{non-empty words})\ w_1 \cdots w_\ell = w \\ M > m_1 > \ldots > m_\ell > 0}} f_{m_1}(w_1)\cdot \ldots \cdot f_{m_\ell}(w_\ell)$$

is a $\mathbb{Q}$-alg. hom.

Proof: Induction on M with $F_{M+1} = f_M \star F_M$. $\square$

$$\sum_{uv=w} f_M(u) F_M(v) = \sum_{u w_2 \cdots w_\ell = w} f_M(u) f_{m_2}(w_1) \cdots f_{m_\ell}(w_\ell)$$

Proposition 19 $\hat{g}^* : \mathfrak{h}^{\mathbb{G}}_* \to \mathbb{Q}[2\pi i][[q]]$ is an $\mathbb{Q}$-alg. hom

Proof: Lemma 18 with $f_m(w) := \psi_w(m\tau)$ and $M \to \infty$. $\square$

Summary:

$$\psi \text{ alg. hom} \overset{\text{Lem. 18}}{\Longrightarrow} \hat{g}^* \text{ alg. hom} \longrightarrow \hat{g}^* \star \mathfrak{h} = \mathbb{G} \text{ alg. hom.}$$

$$\mathfrak{h} \text{ alg. hom} \longrightarrow \hat{g}^* \star \mathfrak{h} = \mathbb{G} \text{ alg. hom.}$$

This can be used to extend the map $\mathbb{G}$ to $\mathfrak{h}'_*$, by using the following

Theorem 20 (Bouillot) There exist an alg. hom.

$$\Psi^*: \mathfrak{h}^1 \longrightarrow \mathcal{O}(\mathbb{H})$$

with $\Psi^*|_{\mathfrak{h}^{\mathcal{G}}} = \Psi$ and $\Psi_{z_1}(\tau) = -\frac{\pi i}{2} - 2\pi i \sum_{n>0} q^n$

$$\left(= \pi \cot(\pi\tau)\right)$$

$$\Psi^* \xrightarrow{\text{Lem. 18}} \hat{g}^*: \mathfrak{h}^0_* \to \mathbb{C}[[q]] \qquad \hat{g}^*: \mathfrak{h}^1_* \to \mathbb{Q}[2\pi i][[q]] \rightsquigarrow \mathcal{G}^*: \mathfrak{h}^1_* \to \mathbb{C}[[q]]$$

$\mathfrak{h}^1_* = \mathfrak{h}^0_*[z_1]$, $\hat{g}^*(z_1) = \hat{g}(1)$, $\mathcal{U}^*$, $\mathcal{G}^* = \hat{g}^* \star \mathcal{U}^*$

Stuffle regularized MES (B.)

By construction: $\mathcal{G}^*|_{\mathfrak{h}^{\mathcal{G}}} = \mathcal{G}$.

What about the shuffle product?

One can equip $\mathfrak{h}^1_{III}$ with the Goncharov coproduct $\Delta_{\mathcal{G}}$.

$\mathcal{I}$ = space of formal iterated integrals

$a_j \in \{0,1\}$ $\quad \mathrm{I}(a_0; a_1, \dots, a_n; a_{n+1})$

$$\left(\mathcal{U}(2,3) = \mathcal{U}(xyxxy) \rightsquigarrow \mathrm{I}(1;01001;0)\right)$$

$$\mathfrak{h}^1_{III} \cong \mathcal{I}^1 := \mathcal{I} \Big/ \mathrm{I}(1;0;0) \hookrightarrow \text{"}-\log(0) := 0\text{"}$$

$$z_{k_1} \cdots z_{k_r} \mapsto \mathrm{I}(1; \underbrace{0\cdots0}_{k_1-1}1 \dots \underbrace{0\cdots0}_{k_r-1}1; 0)$$

Moreover there exist a shuffle version of g:

Proposition 21 (B.-Tasaka) There exists an $\mathbb{Q}$-algebra hom. $\hat{g}^{ш}: \mathfrak{h}^1_{ш} \longrightarrow \mathbb{Q}[2\pi i][[q]]$ with $\hat{g}^{ш} \mid_{\mathfrak{h}^G} = \hat{g}$. ∎

With this one can define shuffle regularized MES (B.-Tasaka)

$$G^{ш} := m \circ (\hat{g}^{ш} \otimes \varphi_{KZ}^{ш}) \circ \Delta_{ш},$$

which are alg. hom. $G^{ш}: \mathfrak{h}^1_{ш} \longrightarrow \mathbb{C}[[q]]$.

Theorem 22 (B.-Tasaka)

We have $G^{ш} \mid_{\mathfrak{h}^G} = G$. ∎

Corollary 23 The MES satisfy the restricted double shuffle relations (RDSH):

$$\forall w, v \in \mathfrak{h}^G \quad G(w * v - w \,ш\, v) = 0$$ ∎

But one can check that $\mathbb{G}$ satisfy more relations, e.g.

$$\mathbb{G}\left(z_2 * z_2 z_1 - z_2 \sqcup\!\sqcup z_2 z_1\right) = 0$$

$$= \mathbb{G}_{4,1} + \mathbb{G}_{2,3} - \mathbb{G}_{2,2,1} - 6\,\mathbb{G}_{3,1,1} \,.$$

Further one can show: $\bullet \in \{\sqcup\!\sqcup, *\}$

$$\mathbb{G}^{\bullet}_3(\tau) - \mathbb{G}^{\bullet}_{2,1}(\tau) = \frac{(2\pi i)^2}{2}\, q\frac{d}{dq}\,\mathbb{G}^{\bullet}_1(\tau)$$

- How to describe these (certainly interesting) relations algebraically?
- How to include derivatives in the algebraic setup?

Calculation:

$$\mathbb{G}^{*}_{2,1}(\tau) = \mathbb{G}^{\sqcup\!\sqcup}_{2,1}(\tau) = \zeta(2,1) + \sum_{m_1,m_2>0} \psi_2(m_1\tau)\,\overbrace{\psi_1(m_2\tau)}^{(2\pi i)\left(\frac{1}{2}+\sum_{n>0} q^{m_2 n}\right)}$$

$$= \zeta(2,1) + \hat{g}(2,1) + \frac{(2\pi i)^3}{2}\sum_{m>0}\underbrace{\frac{(m-1)\,q^m}{(1-q^m)^2}}_{q\frac{d}{dq}g(1)-g(2)}$$

One can show:

$$g(3) - \left(g(2,1) - \tfrac{1}{2}g(2)\right) = q\frac{d}{dq}\,g(1)$$