

Combinatorial multiple Eisenstein series

Henrik Bachmann

Nagoya University

based on joint works (in progress) with

U. Kühn (Hamburg), N. Matthes (Oxford)

A. Burmester (Hamburg)

$$G_{k_1, \dots, k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}$$
$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} B\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} + B\begin{pmatrix} X_2, X_1 \\ Y_2, Y_1 \end{pmatrix} + \frac{B\begin{pmatrix} X_1 \\ Y_1+Y_2 \end{pmatrix} - B\begin{pmatrix} X_2 \\ Y_1+Y_2 \end{pmatrix}}{X_1 - X_2}$$
$$B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = B\begin{pmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{pmatrix}$$

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www.henrikbachmann.com

Overview

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Multiple zeta values
&
double shuffle relations

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(analytic)
Multiple Eisenstein series

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Symmetrization
&
 q -double shuffle relations

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Combinatorial
multiple Eisenstein series

① MZV & DSH - Definition

Definition

For $k_1 \geq 2, k_2, \dots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}.$$

(Ohsuruf order)

By r we denote its **depth** and $k_1 + \dots + k_r$ will be called its **weight**.

- \mathcal{Z} : \mathbb{Q} -algebra of MZVs
- \mathcal{Z}_k : \mathbb{Q} -vector space of MZVs of weight k .

① MZV & DSH - Harmonic/stuffle & shuffle product

There are two different ways to express the product of MZV in terms of MZV.

Harmonic/stuffle product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1) \cdot \zeta(k_2) &= \sum_{m>0} \frac{1}{m^{k_1}} \sum_{n>0} \frac{1}{n^{k_2}} \\ &= \sum_{0 < m < n} \frac{1}{m^{k_1} n^{k_2}} + \sum_{0 < n < m} \frac{1}{m^{k_1} n^{k_2}} + \sum_{m=n>0} \frac{1}{m^{k_1+k_2}} \\ &= \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1) \cdot \zeta(k_2) = \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j).$$

① MZV & DSH - Double shuffle relations

These two product expressions give various \mathbb{Q} -linear relations between MZV.

Example

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &\stackrel{\text{harmonic}}{=} \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &\stackrel{\text{shuffle}}{=} \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1). \\ \implies 2\zeta(3, 2) + 6\zeta(4, 1) &\stackrel{\text{double shuffle}}{=} \zeta(5).\end{aligned}$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0} \frac{1}{m^2 n} = \zeta(2, 1) = \zeta(3) = \sum_{m>0} \frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations.

① MZV & DSH - Regularization

Regularization

There are two ways (**harmonic/stuffle** $(*)$ and **shuffle** (\sqcup) **regularization**) to extend the definition of $\zeta(k_1, \dots, k_r)$ to all $k_1, \dots, k_r \geq 1$.

Example:

depth 1: $\zeta^*(1) = \zeta^\sqcup(1) = 0$ and $\zeta^*(k) = \zeta^\sqcup(k) = \zeta(k)$ for $k > 1$.

depth 2:

$$k_1 \geq 2, k_2 \geq 1 : \quad \zeta^*(k_1, k_2) = \zeta^\sqcup(k_1, k_2) = \zeta(k_1, k_2),$$

$$k_1 = 1, k_2 > 1 : \quad \zeta^*(1, k_2) = \zeta^\sqcup(1, k_2) = -\zeta(k_2, 1) - \zeta(k_2 + 1),$$

$$k_1 = 1, k_2 = 1 : \quad \zeta^*(1, 1) = -\frac{1}{2}\zeta(2), \quad \zeta^\sqcup(1, 1) = 0.$$

$$\zeta^\sqcup(k_1, k_2) = \zeta^*(k_1, k_2) + \delta_{k_1, 1} \delta_{k_2, 1} \frac{1}{2} \zeta(2).$$

① MZV & DSH - Regularized double shuffle

For all $k_1, k_2 \geq 1$ we have

$$\begin{aligned}\zeta^*(k_1) \cdot \zeta^*(k_2) &= \zeta^*(k_1, k_2) + \zeta^*(k_2, k_1) + \zeta^*(k_1 + k_2) \\ &= \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta^{\bowtie}(j, k_1 + k_2 - j).\end{aligned}$$

① MZV & DSH - Regularized double shuffle

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$$\begin{aligned}\zeta^*(k_1) \cdot \zeta^*(k_2) &= \zeta^*(k_1, k_2) + \zeta^*(k_2, k_1) + \zeta^*(k_1 + k_2) \\ &= \sum_{j=1}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta^{\square}(j, k_1 + k_2 - j).\end{aligned}$$

Since $\zeta^{\square}(k_1, k_2) = \zeta^*(k_1, k_2) + \delta_{k_1, 1} \delta_{k_2, 1} \frac{1}{2} \zeta(2)$ the generating series

$$Z^*(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} \zeta^*(k_1, \dots, k_r) X_1^{k_1-1} \dots X_r^{k_r-1}$$

satisfies

$$\begin{aligned}Z^*(X_1) Z^*(X_2) &= Z^*(X_1, X_2) + Z^*(X_2, X_1) + \frac{Z^*(X_1) - Z^*(X_2)}{X_1 - X_2} \\ &= Z^*(X_1 + X_2, X_2) + Z^*(X_1 + X_2, X_1) + \zeta(2).\end{aligned}$$

① MZV & DSH - General dsh relations

- A : \mathbb{Q} -algebra.
- For $z_{k_1, \dots, k_r} \in A$ for $k_1, \dots, k_r \geq 1$ we write

$$Z(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} z_{k_1, \dots, k_r} X_1^{k_1-1} \cdots X_r^{k_r-1}.$$

- A collection $Z = (Z(X_1, \dots, X_r))_{r \geq 1}$ will be called a **mould**.
(Please ask Komiyama-san for the correct definition of a mould)

Definition

A mould Z satisfies the **double shuffle relations** (in depth 2) if

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

① MZV & DSH - Known solutions to the double shuffle relations

- $A = \mathbb{R}$: Harmonic regularized multiple zeta values

$$z_{k_1, \dots, k_r} = \zeta^*(k_1, \dots, k_r).$$

- $A = \mathbb{Q}$: Explicit solutions are known up to depth 3 (Brown, Ecalle, Gangl-Kaneko-Zagier, Tasaka). In depth 1 they are all given by

$$z_k = \begin{cases} -\frac{B_k}{2k!} = \frac{\zeta(k)}{(2\pi i)^k}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

- $A = \mathbb{Q}$: Solution exist in all depths (Drinfel'd + Furusho, Racinet).

① MZV & DSH - Can we lift these relations?

The Riemann zeta values are the constant term of Eisensteins series

$$G_k = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1-q^n} \in \mathbb{C}[[q]]$$

and $-\frac{B_k}{2k!}$ is the constant term of its normalized version ($k \geq 2$ even)

$$\tilde{G}_k := (2\pi i)^{-k} G_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1-q^n} \in \mathbb{Q}[[q]].$$

Natural questions

- What about the case $A = \mathbb{C}[[q]]$ of $A = \mathbb{Q}[[q]]$?
- Are there solutions to the double shuffle equations with $z_k = G_k$ or $z_k = \tilde{G}_k$?
- Is there a depth two Eisenstein series?

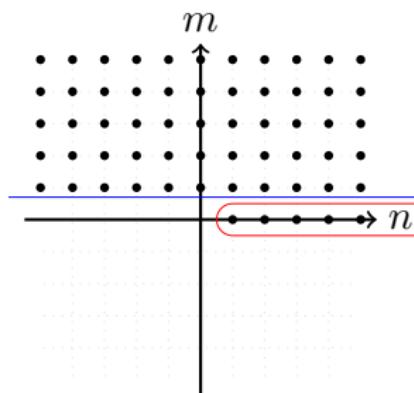
② Multiple Eisenstein series - An order on lattices

Let $\tau \in \mathbb{H}$. We define an order \succ on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ by setting

$$\lambda_1 \succ \lambda_2 : \Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for $\lambda_1, \lambda_2 \in \mathbb{Z}\tau + \mathbb{Z}$ and the following set of positive lattice points

$$P := \{m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R.$$



In other words: $\lambda_1 \succ \lambda_2$ iff λ_1 is above or on the right of λ_2 .

② Multiple Eisenstein series - Multiple Eisenstein series

Definition

For integers $k_1 \geq 3, k_2, \dots, k_r \geq 2$, we define the **multiple Eisenstein series** by

$$G_{k_1, \dots, k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the **harmonic product**, i.e. it is for example

$$G_2(\tau) \cdot G_3(\tau) = G_{2,3}(\tau) + G_{3,2}(\tau) + G_5(\tau).$$

② Multiple Eisenstein series - Classical Eisenstein series

In depth one we have for $k \geq 3$

$$G_k(\tau) = \sum_{\substack{\lambda \in \mathbb{Z}\tau + \mathbb{Z} \\ \lambda > 0}} \frac{1}{\lambda^k} = \sum_{\substack{m > 0 \\ \vee (m=0 \wedge n>0)}} \frac{1}{(m\tau + n)^k} = \zeta(k) + \sum_{m>0} \underbrace{\sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}}_{=: \Psi_k(m\tau)}$$

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We refer to $\Psi_k(\tau)$ as the **monotangent function**, which satisfies for $k \geq 2$

$$\Psi_k(\tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d>0} d^{k-1} q^d \quad (q = e^{2\pi i \tau}).$$

This gives

$$G_k(\tau) = \zeta(k) + \sum_{m>0} \Psi_k(m\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{m>0 \\ d>0}} d^{k-1} q^{md}.$$

② Multiple Eisenstein series - q -MZV

Definition

For $k_1, \dots, k_r \geq 1$ we define the/a **q -analogue of multiple zeta values** by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ d_1, \dots, d_r > 0}} \frac{d_1^{k_1-1}}{(k_1-1)!} \cdots \frac{d_r^{k_r-1}}{(k_r-1)!} q^{m_1 d_1 + \dots + m_r d_r} \in \mathbb{Q}[[q]].$$

Proposition

- For $k_1, \dots, k_r \geq 2$ we have ($q = e^{2\pi i \tau}$)

$$g(k_1, \dots, k_r) = (-2\pi i)^{-(k_1 + \dots + k_r)} \sum_{m_1 > \dots > m_r > 0} \Psi_{k_1}(m_1 \tau) \dots \Psi_{k_r}(m_r \tau).$$

- For $k_1 \geq 2, k_2, \dots, k_r \geq 1$ we have

$$\lim_{q \rightarrow 1} (1-q)^{k_1 + \dots + k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r).$$

② Multiple Eisenstein series - Fourier expansion

Theorem (B., 2012)

The multiple Eisenstein series $G_{k_1, \dots, k_r}(\tau)$ have a Fourier expansion of the form

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n>0} a_n q^n \quad (q = e^{2\pi i \tau})$$

and they can be written explicitly as a $\mathcal{Z}[2\pi i]$ -linear combination of q -analogues of multiple zeta values g . In particular, $a_n \in \mathcal{Z}[2\pi i]$.

Examples

$$G_k(\tau) = \zeta(k) + (-2\pi i)^k g(k),$$

$$G_{3,2}(q) = \zeta(3, 2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3, 2).$$

② Multiple Eisenstein series - Fourier expansion construction

Summing over $\lambda_1 \succ \cdots \succ \lambda_r \succ 0$ is by definition equivalent to summing over all $\lambda_1, \dots, \lambda_r$ with

$$\lambda_{i-1} - \lambda_i \in P = \textcolor{blue}{U} \cup \textcolor{red}{R} \quad (\lambda_0 := 0).$$

Since $\lambda_{i-1} - \lambda_i$ can be either in $\textcolor{blue}{U}$ or in $\textcolor{red}{R}$ we can split up the sum in the definition of the MES into 2^r terms. For $w_1, \dots, w_r \in \{\textcolor{blue}{U}, \textcolor{red}{R}\}$ we define

$$G_{n_1, \dots, n_r}^{w_1 \dots w_r}(\tau) = \sum_{\substack{\lambda_1, \dots, \lambda_r \in \Lambda_\tau \\ \lambda_{i-1} - \lambda_i \in w_i}} \frac{1}{\lambda_1^{n_1} \cdots \lambda_r^{n_r}}.$$

With this we get

$$G_{n_1, \dots, n_r}(\tau) = \sum_{w_1, \dots, w_r \in \{U, R\}} G_{n_1, \dots, n_r}^{w_1 \dots w_r}(\tau).$$

② Multiple Eisenstein series - Fourier expansion in depth two

In depth two the $2^2 = 4$ terms are given by

$$G_{k_1, k_2}^{RR}(\tau) = \sum_{\substack{m_1=m_2=0 \\ n_1 > n_2 > 0}} \frac{1}{(m_1\tau + n_1)^{k_1} (m_2\tau + n_2)^{k_2}} = \zeta(k_1, k_2),$$

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$$G_{k_1, k_2}^{UR}(\tau) = \sum_{\substack{m_1>m_2=0 \\ n_1 \in \mathbb{Z}, n_2>0}} \frac{1}{(m_1\tau+n_1)^{k_1}(n_2)^{k_2}} = \sum_{m>0} \Psi_{k_1}(m\tau) \zeta(k_2),$$

② Multiple Eisenstein series - Fourier expansion in depth two

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$$G_{k_1, k_2}^{UU}(\tau) = \sum_{\substack{m_1 > m_2 > 0 \\ n_1, n_2 \in \mathbb{Z}}} \frac{1}{(m_1\tau + n_1)^{k_1} (m_2\tau + n_2)^{k_2}} = \sum_{m_1 > m_2 > 0} \Psi_{k_1}(m_1\tau) \Psi_{k_2}(m_2\tau),$$

② Multiple Eisenstein series - Fourier expansion in depth two

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$$G_{k_1, k_2}^{UR}(\tau) = \sum_{\substack{m_1 > m_2=0 \\ n_1 \in \mathbb{Z}, n_2 > 0}} \frac{1}{(m_1\tau + n_1)^{k_1} (n_2)^{k_2}} = \sum_{m>0} \Psi_{k_1}(m\tau) \zeta(k_2),$$

$$G_{k_1, k_2}^{UU}(\tau) = \sum_{\substack{m_1 > m_2 > 0 \\ n_1, n_2 \in \mathbb{Z}}} \frac{1}{(m_1\tau + n_1)^{k_1} (m_2\tau + n_2)^{k_2}} = \sum_{m_1 > m_2 > 0} \Psi_{k_1}(m_1\tau) \Psi_{k_2}(m_2\tau),$$

$$G_{k_1, k_2}^{RU}(\tau) = \sum_{\substack{m_1=m_2>0 \\ n_1 > n_2 \\ n_1, n_2 \in \mathbb{Z}}} \frac{1}{(m_1\tau + n_1)^{k_1} (m_2\tau + n_2)^{k_2}}$$

$$= \underbrace{\sum_{m>0} \sum_{\substack{n_1 > n_2 \\ n_1, n_2 \in \mathbb{Z}}} \frac{1}{(m\tau + n_1)^{k_1} (m\tau + n_2)^{k_2}}}_{=: \Psi_{k_1, k_2}(m\tau)} =: \sum_{m>0} \Psi_{k_1, k_2}(m\tau) = ?? \text{ q-MZV } ??$$

② Multiple Eisenstein series - Multitangent functions

Definition

For $k_1, \dots, k_r \geq 2$ and $\tau \in \mathbb{H}$ define the **multitangent function** by

$$\Psi_{k_1, \dots, k_r}(\tau) := \sum_{\substack{n_1 > \dots > n_r \\ n_i \in \mathbb{Z}}} \frac{1}{(\tau + n_1)^{k_1} \cdots (\tau + n_r)^{k_r}}.$$

Theorem (Bouillot 2011, B. 2012)

For $k_1, \dots, k_r \geq 2$ and $K = k_1 + \dots + k_r$ the multitangent function can be written as

$$\Psi_{k_1, \dots, k_r}(\tau) = \sum_{j=2}^K \alpha_{K-j} \Psi_j(\tau)$$

with $\alpha_{K-j} \in \mathcal{Z}_{K-j}$.

Proof idea: Use partial fraction decomposition.

② Multiple Eisenstein series - Multitangent \rightsquigarrow Monotangent

Example: Reduction of multitangent to monotangent

$$\begin{aligned}\Psi_{3,2}(\tau) &= \sum_{n_1 > n_2} \frac{1}{(\tau + n_1)^3(\tau + n_2)^2} \\&= \sum_{n_1 > n_2} \left(\frac{1}{(n_1 - n_2)^3(\tau + n_2)^2} - \frac{3}{(n_1 - n_2)^4(\tau + n_2)} \right) + \\&\quad \sum_{n_1 > n_2} \left(\frac{1}{(n_1 - n_2)^2(\tau + n_1)^3} + \frac{2}{(n_1 - n_2)^3(\tau + n_1)^2} + \frac{3}{(n_1 - n_2)^4(\tau + n_1)} \right) \\&= 3\zeta(3)\Psi_2(\tau) + \zeta(2)\Psi_3(\tau).\end{aligned}$$

② Multiple Eisenstein series - Fourier expansion of $G_{3,2}$

$$\Psi_{3,2}(\tau) = 3\zeta(3)\Psi_2(\tau) + \zeta(2)\Psi_3(\tau).$$

Example: Fourier expansion of $G_{3,2}$

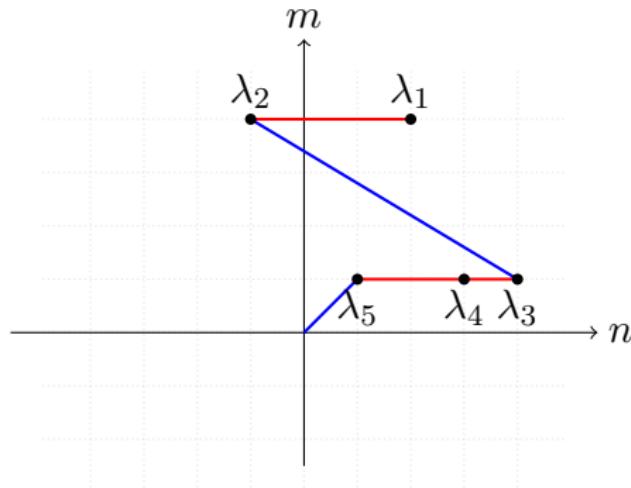
$$\begin{aligned} G_{3,2}(\tau) &= G_{3,2}^{RR}(\tau) + G_{3,2}^{UR}(\tau) + G_{3,2}^{RU}(\tau) + G_{3,2}^{UU}(\tau) \\ &= \zeta(3, 2) + \sum_{m>0} \Psi_3(m\tau)\zeta(2) + \sum_{m>0} \Psi_{3,2}(m\tau) + \sum_{m_1>m_2>0} \Psi_3(m_1\tau)\Psi_2(m_1\tau) \\ &= \zeta(3, 2) + \sum_{m>0} (3\zeta(3)\Psi_2(m\tau) + 2\zeta(2)\Psi_3(m\tau)) + \sum_{m_1>m_2>0} \Psi_3(m_1\tau)\Psi_2(m_1\tau). \end{aligned}$$

With this we get the Fourier expansion of $G_{2,3}$:

$$G_{3,2}(q) = \zeta(3, 2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3, 2).$$

② Multiple Eisenstein series - Appearance of Multitangent in general

Example: $w_1 w_2 w_3 w_4 w_5 = RURRU$



A summand of $G_{k_1, k_2, k_3, k_4, k_5}^{RURRU}$.

By definition of the multitangent functions we can write

$$G_{k_1, k_2, k_3, k_4, k_5}^{RURRU}(\tau) = \sum_{m_1 > m_2 > 0} \Psi_{k_1, k_2}(m_1 \tau) \Psi_{k_3, k_4, k_5}(l_2 \tau).$$

Calculation of the Fourier expansion of multiple Eisenstein series

$$G_{k_1, \dots, k_r}(\tau) = \sum_{\substack{\lambda_1 > \dots > \lambda_r > 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}$$

$$G_{n_1, \dots, n_r}(\tau) = \sum_{w_1, \dots, w_r \in \{U, R\}} G_{n_1, \dots, n_r}^{w_1 \dots w_r}(\tau)$$

MZV

$$\zeta(k_1, \dots, k_r)$$

Sums of multitangent functions

$$\sum_{m_1 > m_2 > 0} \Psi_{k_1, k_2}(m_1 \tau) \Psi_{k_3, k_4, k_5}(l_2 \tau)$$

Reduction multitangent to monotangent

$$\Psi_{k_1, \dots, k_r}(\tau) = \sum_{j=2}^K \alpha_{K-j} \Psi_j(\tau)$$

q-MZV (sums of monotangent functions)

$$g(k_1, \dots, k_r) = (-2\pi i)^{-(k_1 + \dots + k_r)} \sum_{m_1 > \dots > m_r > 0} \Psi_{k_1}(m_1 \tau) \dots \Psi_{k_r}(m_r \tau)$$

② Multiple Eisenstein series - Do they satisfy double shuffle?

There are different ways to extend the definition of G_{k_1, \dots, k_r} to $k_1, \dots, k_r \geq 1$

- Formal double zeta space realization $G_{r,s}$ (Gangl-Kaneko-Zagier, 2006)

$$\begin{aligned} G_{k_1} \cdot G_{k_2} + (\delta_{k_1,2} + \delta_{k_2,2}) \frac{G'_{k_1+k_2-2}}{2(k_1+k_2-2)} &= G_{k_1,k_2} + G_{k_2,k_1} + G_{k_1+k_2} \\ &= \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) G_{j,k_1+k_2-j}, \quad (k_1+k_2 \geq 3). \end{aligned}$$

- Finite double shuffle version $G_{r,s}$ (Kaneko, 2007).
- Shuffle regularized multiple Eisenstein series $G_{k_1, \dots, k_r}^{\sqcup\sqcup}$ (B.-Tasaka, 2017)
- Harmonic regularized multiple Eisenstein series G_{k_1, \dots, k_r}^* (B., 2019)

Observation

- No version of these objects satisfy the double shuffle relations for all indices/weights.
- The derivative is always somewhere as an extra term.

③ Symmetril, Swap & q-DSH - General Idea

General idea

- Include also (arbitrary) derivatives as objects.
- Instead of series $Z(X_1, \dots, X_r)$ we will consider generating series with two types of variables X_i and Y_i .
- Roughly: X_i : weight, Y_i : derivative.
- In the case $Y_i = 0$, we get back our original story.

A : \mathbb{Q} -algebra

$$B\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} \in A[[X_1, Y_1, \dots, X_r, Y_r]].$$

Definition

A collection $B = \left(B\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} \right)_{r \geq 1}$ will be called a **bi-mould**.

③ Symmetril, Swap & q-DSH - Symmetril

Definition

A bimould B is **symmetril** (up to depth 3), if

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2},$$

$$\begin{aligned} B\binom{X_1}{Y_1}B\binom{X_2, X_3}{Y_2, Y_3} &= B\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} + B\binom{X_2, X_1, X_3}{Y_2, Y_1, Y_3} + B\binom{X_2, X_3, X_1}{Y_2, Y_3, Y_1} \\ &+ \frac{B\binom{X_1, X_3}{Y_1+Y_2, Y_3} - B\binom{X_2, X_3}{Y_1+Y_2, Y_3}}{X_1 - X_2} + \frac{B\binom{X_2, X_1}{Y_2, Y_1+Y_3} - B\binom{X_2, X_3}{Y_2, Y_1+Y_3}}{X_1 - X_3}. \end{aligned}$$

Remark

This product corresponds to the harmonic product for the coefficients of B .

③ Symmetril, Swap & q-DSH - Swap

Definition

A bimould B is called **swap invariant** if

$$B\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = B\begin{pmatrix} Y_1 + \dots + Y_r, Y_1 + \dots + Y_{r-1}, \dots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \dots, X_2 - X_3, X_1 - X_2 \end{pmatrix}.$$

$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}, \quad B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} \stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{pmatrix}.$$

③ Symmetril, Swap & q-DSH - q-shuffle

Recall **symmetrility** and **swap** in depth 2

$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}, \quad B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} \stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{pmatrix},$$

$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} B\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \stackrel{\text{IL}}{=} B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} + B\begin{pmatrix} X_2, X_1 \\ Y_2, Y_1 \end{pmatrix} + \frac{B\begin{pmatrix} X_1 \\ Y_1+Y_2 \end{pmatrix} - B\begin{pmatrix} X_2 \\ Y_1+Y_2 \end{pmatrix}}{X_1 - X_2}.$$

Definition

Swap + Symmtril + Swap = **q-shuffle**

$$\begin{aligned} B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} B\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} &\stackrel{\text{sw}}{=} B\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix} B\begin{pmatrix} Y_2 \\ X_2 \end{pmatrix} \\ &\stackrel{\text{IL}}{=} B\begin{pmatrix} Y_1, Y_2 \\ X_1, X_2 \end{pmatrix} + B\begin{pmatrix} Y_2, Y_1 \\ X_2, X_1 \end{pmatrix} + \frac{B\begin{pmatrix} Y_1 \\ X_1+X_2 \end{pmatrix} - B\begin{pmatrix} Y_2 \\ X_1+X_2 \end{pmatrix}}{Y_1 - Y_2} \\ &\stackrel{\text{sw}}{=} B\begin{pmatrix} X_1 + X_2, X_1 \\ Y_2, Y_1 - Y_2 \end{pmatrix} + B\begin{pmatrix} X_1 + X_2, X_2 \\ Y_1, Y_2 - Y_1 \end{pmatrix} + \frac{B\begin{pmatrix} X_1 + X_2 \\ Y_1 \end{pmatrix} - B\begin{pmatrix} X_1 + X_2 \\ Y_2 \end{pmatrix}}{Y_1 - Y_2}. \end{aligned}$$

» This interpretation of the shuffle product will also appear in Jan-Willems talk in Toyama.

③ Symmetril, Swap & q-DSH - q-double shuffle

Definition

A bimould satisfies **q-double shuffle** (in depth 2) if

$$\begin{aligned} B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} &= B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2} \\ &= B\binom{X_1 + X_2, X_1}{Y_2, Y_1 - Y_2} + B\binom{X_1 + X_2, X_2}{Y_1, Y_2 - Y_1} + \frac{B\binom{X_1 + X_2}{Y_1} - B\binom{X_1 + X_2}{Y_2}}{Y_1 - Y_2}, \end{aligned}$$

i.e. B is symmetril and satisfies the q -shuffle product formula.

- Clearly: Symmetril + Swap invariant \implies q-double shuffle.
- Compare this to the double shuffle relations

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

③ Symmetril, Swap & q-DSH -

Solution to q-dsh 0 => solution to dsh

Proposition

Let B be **symmetril** and **swap invariant** with $b\binom{k}{1} = 0$ for $k > 0$. Then

$$Z(X) = B\binom{X}{0}, \quad Z(X_1, X_2) = B\binom{X_1, X_2}{0, 0}$$

satisfies the double shuffle relations.

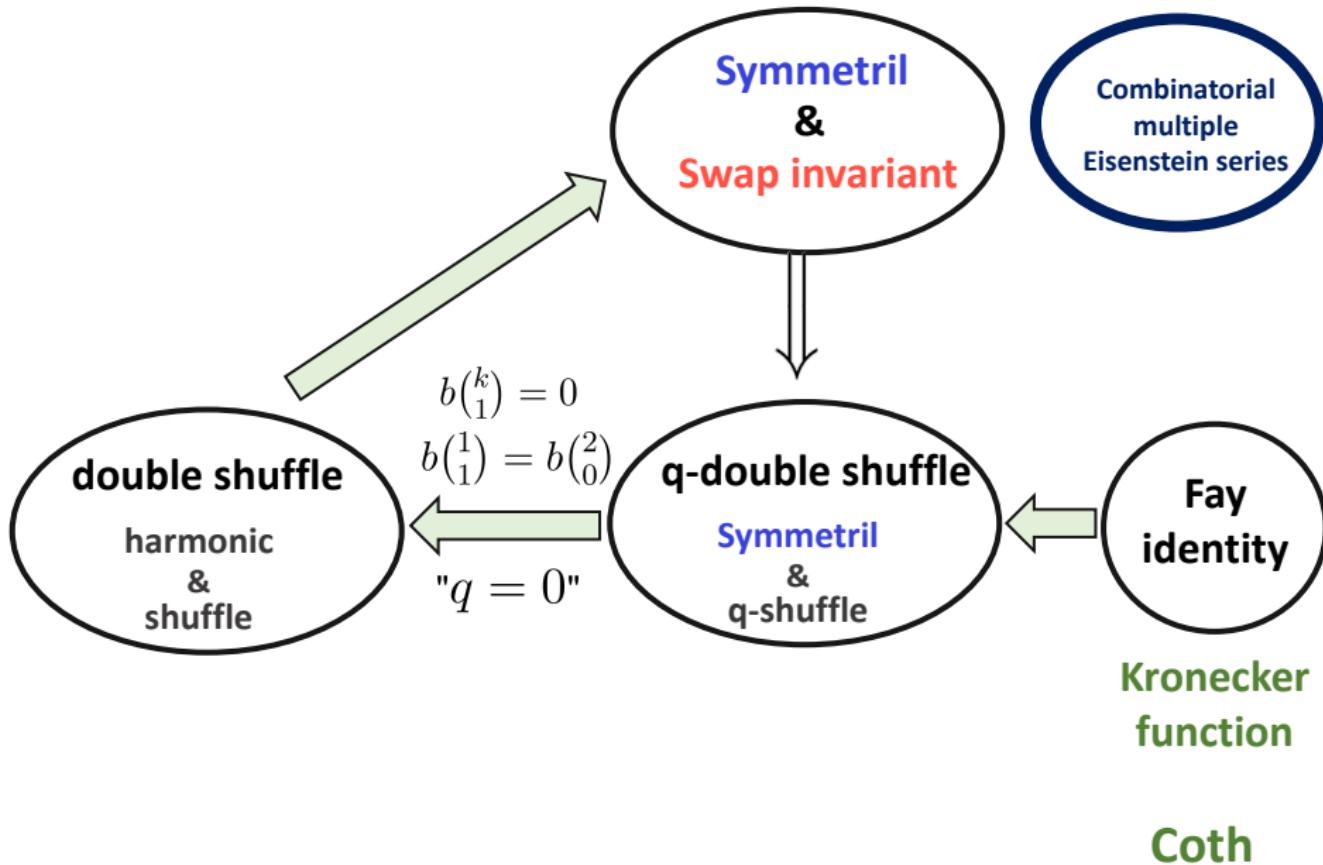
Proof:

$$\begin{aligned} \frac{B\binom{X_1+X_2}{Y_1} - B\binom{X_1+X_2}{Y_2}}{Y_1 - Y_2} &\Big|_{Y_1=Y_2=0} = \sum_{k \geq 1} b\binom{k}{1} (X_1 + X_2)^{k-1} \\ &= b\binom{1}{1} \stackrel{\text{SW}}{=} b\binom{2}{0} = z_2. \end{aligned}$$

③ Symmetril, Swap & q-DSH - Finding solutions

How to get solutions to swap & symmetril?

- Solutions to double shuffle give solutions (depth ≤ 3)
- Fay Identity (up to depth ≤ 2)
- Combinatorial MES (depth ≤ 3)



③ Symmetril, Swap & q-DSH -

Proposition

Let Z satisfy the double shuffle relations. Then

$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = Z(X_1) + Z(Y_1),$$

$$B\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = Z(X_1, X_2) + Z(Y_1 + Y_2, Y_1) + Z(X_2)Z(Y_1) + \frac{1}{2}z_2$$

is **symmetril** and **swap invariant**

Remark

An analogue statement also exists for depth 3.

③ Symmetril, Swap & q-DSH - Fay Identity

For a $B \in A[[X_1, Y_1]]$ we write

$$B^{\text{ev}}\binom{X}{Y} = \frac{1}{2} \left(B\binom{X}{Y} - B\binom{-X}{-Y} \right).$$

Proposition

If B satisfies the q -double shuffle relations in depth 2, then

$$F\binom{X}{Y} = -\frac{1}{2} \left(\frac{1}{X} + \frac{1}{Y} \right) + B^{\text{ev}}\binom{X}{Y}$$

satisfies the **Fay identity**

$$F\binom{X_1}{Y_1} F\binom{X_2}{Y_2} + F\binom{X_1 - X_2}{-Y_2} F\binom{X_1}{Y_1 + Y_2} + F\binom{-X_2}{-Y_1 - Y_2} F\binom{X_1 - X_2}{Y_1} = 0.$$

③ Symmetril, Swap & q-DSH - Fay Identity \implies q-DSH

The converse is also true:

Theorem (B.-Kühn-Matthes, 2020+) (rough version)

Any odd Laurent series $F \in A((X, Y))$ of the form

$$F\left(\frac{X}{Y}\right) = -\frac{1}{2} \left(\frac{1}{X} + \frac{1}{Y} \right) + B\left(\frac{X}{Y}\right),$$

which satisfies the Fay Identity, gives an (explicit) solution to q-double shuffle in depth 2.

Fact

The **Kronecker function** defined by

$$K_q\left(\frac{X}{Y}\right) = -\frac{1}{2} \left(\frac{1}{X} + \frac{1}{Y} \right) + \sum_{\substack{r,s \geq 0 \\ r+s \text{ odd}}} \frac{|r-s|!}{r! s!} \left(q \frac{d}{dq} \right)^{\min\{r,s\}} \tilde{G}_{|r-s|+1}(q) X^r Y^s$$

satisfies the Fay identity.

③ Symmetril, Swap & q-DSH - Kronecker solution to q-dsh

- Theorem + Kronecker function $K_q\left(\frac{X}{Y}\right) \implies$ explicit solution to q -double shuffle.
- This solution has the **even** Eisenstein series and their derivatives as objects in depth one.

In the special case $q = 0$ we get an explicit rational solution to q -double shuffle.

Proposition

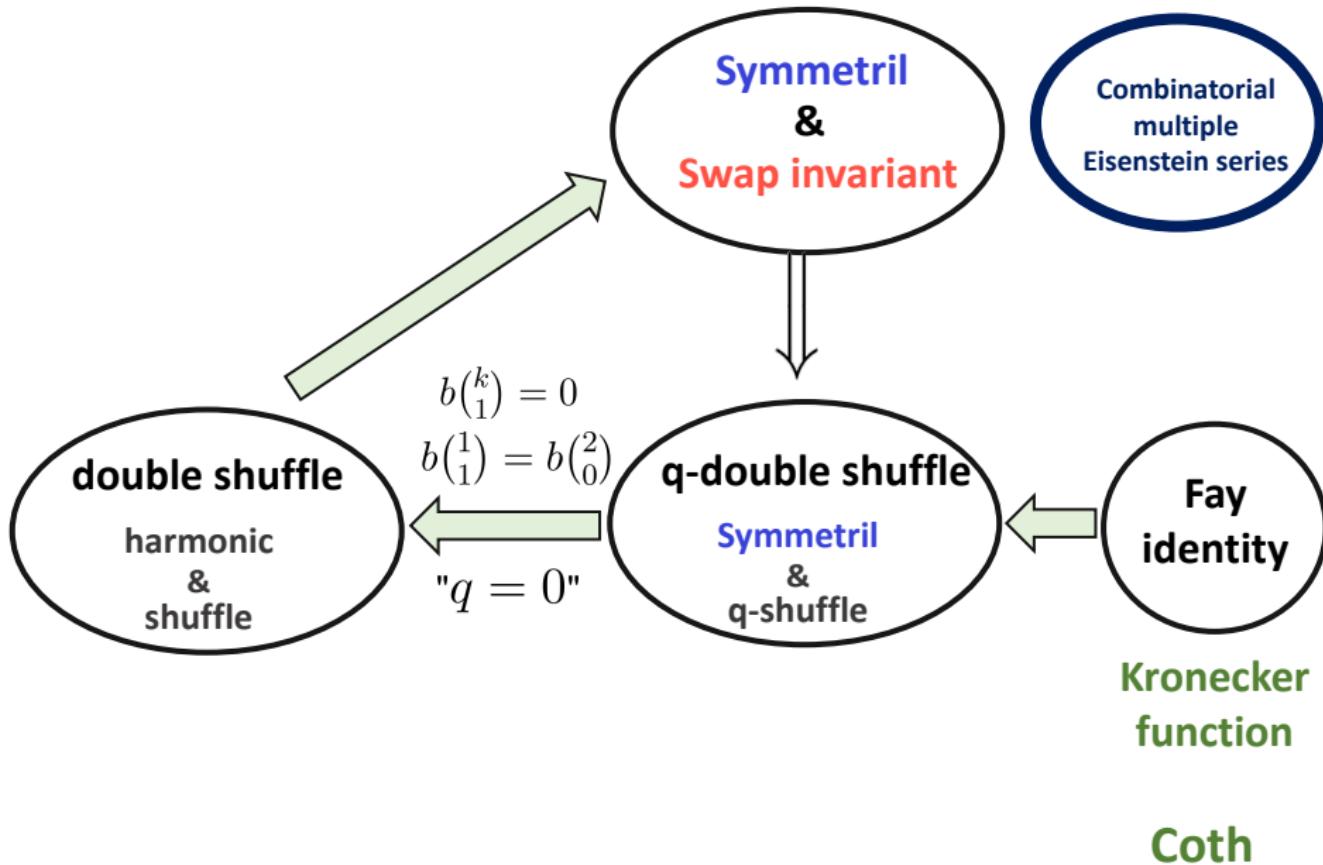
There exist a (non-unique) **symmetril** and **swap invariant** bimould β with rational coefficients.
depth 1:

$$\beta\left(\frac{X}{Y}\right) = \frac{1}{2} \left(\frac{1}{X} + \frac{1}{Y} \right) - \frac{1}{4} \left(\coth\left(\frac{X}{2}\right) + \coth\left(\frac{Y}{2}\right) \right).$$

depth 2: Explicit formula coming from Theorem + Kronecker.

In the following we fix one of these possible bi-moulds

$$\beta\left(\frac{X_1, \dots, X_r}{Y_1, \dots, Y_r}\right) \in \mathbb{Q}[[X_1, Y_1, \dots, X_r, Y_r]].$$



Calculation of the Fourier expansion of multiple Eisenstein series

$$G_{k_1, \dots, k_r}(\tau) = \sum_{\substack{\lambda_1 > \dots > \lambda_r > 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}$$

$$G_{n_1, \dots, n_r}(\tau) = \sum_{w_1, \dots, w_r \in \{U, R\}} G_{n_1, \dots, n_r}^{w_1 \dots w_r}(\tau)$$

MZV

$$\zeta(k_1, \dots, k_r)$$

Sums of multitangent functions

$$\sum_{m_1 > m_2 > 0} \Psi_{k_1, k_2}(m_1 \tau) \Psi_{k_3, k_4, k_5}(l_2 \tau)$$

Reduction multitangent to monotangent

$$\Psi_{k_1, \dots, k_r}(\tau) = \sum_{j=2}^K \alpha_{K-j} \Psi_j(\tau)$$

q-MZV (sums of monotangent functions)

$$g(k_1, \dots, k_r) = (-2\pi i)^{-(k_1 + \dots + k_r)} \sum_{m_1 > \dots > m_r > 0} \Psi_{k_1}(m_1 \tau) \dots \Psi_{k_r}(m_r \tau)$$

Construction of combinatorial multiple Eisenstein series

$$G\binom{k_1, \dots, k_r}{d_1, \dots, d_r} \quad \mathfrak{G}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r}$$

Rational solution to q-dsh

$$\beta\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r}$$

Sums of multiple version of L

$$\sum_{m_1 > m_2 > 0} L_{m_1}\binom{X_1, X_2}{Y_1, Y_2} L_{m_2}\binom{X_3}{Y_3}$$

Define multiple version of L by single version of L

The series g (sums of single version of L)

$$\mathfrak{g}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{m_1 > \dots > m_r > 0} L_{m_1}\binom{X_1}{Y_1} \dots L_{m_r}\binom{X_r}{Y_r}$$

④ Combinatorial MES - The bimould \mathfrak{g}

Define for $m \geq 1$ the series

$$L_m \binom{X}{Y} = \frac{e^{X+mY} q^m}{1 - e^X q^m} = \sum_{n \geq 1} e^{nX+mY} q^{mn}.$$

Definition

We define the bimould \mathfrak{g} for all depth $r \geq 1$ by

$$\mathfrak{g} \binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{m_1 > \dots > m_r > 0} L_{m_1} \binom{X_1}{Y_1} \dots L_{m_r} \binom{X_r}{Y_r}.$$

Proposition

The bimould \mathfrak{g} is swap invariant.

④ Combinatorial MES - The bimould \mathfrak{g}

Example: Swap invariance of \mathfrak{g} in depth 2

$$\begin{aligned}\mathfrak{g} \begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} &= \sum_{m_1 > m_2 > 0} L_{m_1} \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} L_{m_2} \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} \\ &= \sum_{\substack{m_1 > m_2 > 0 \\ n_1, n_2 > 0}} e^{n_1 X_1 + n_2 X_2 + m_1 Y_1 + m_2 Y_2} q^{m_1 n_1 + m_2 n_2} = (\star)\end{aligned}$$

Change of variables \longleftrightarrow swap of variables

$$\left\{ \begin{array}{lcl} m_1 = m'_1 + m'_2 & , & m_2 = m'_1 \\ n_1 = n'_2 & , & n_2 = n'_1 - n'_2 \end{array} \right\} \implies m_1 n_1 + m_2 n_2 = m'_1 n'_1 + m'_2 n'_2.$$

$$\begin{aligned}(\star) &= \sum_{\substack{m'_1, m'_2 > 0 \\ n'_1 > n'_2 > 0}} e^{n'_2 X_1 + (n'_1 - n'_2) X_2 + (m'_1 + m'_2) Y_1 + m'_1 Y_2} q^{m'_1 n'_1 + m'_2 n'_2} \\ &= \mathfrak{g} \begin{pmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{pmatrix}.\end{aligned}$$

④ Combinatorial MES - Product of \mathfrak{g}

$$\begin{aligned}\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2}{Y_2} &= \sum_{m_1>0} L_{m_1}\binom{X_1}{Y_1} \sum_{m_2>0} L_{m_2}\binom{X_2}{Y_2} \\ &= \left(\sum_{m_1>m_2>0} + \sum_{m_2>m_1>0} + \sum_{m_1=m_2>0} \right) L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} \\ &= \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1} + \sum_{m>0} L_m\binom{X_1}{Y_1} L_m\binom{X_2}{Y_2}.\end{aligned}$$

To describe the product of \mathfrak{g} we need to describe for a fixed m the product of L_m .

④ Combinatorial MES - Product of \mathfrak{g}

$$\begin{aligned}
 \mathfrak{g}\binom{X_1}{Y_1} \mathfrak{g}\binom{X_2}{Y_2} &= \sum_{m_1 > 0} L_{m_1} \binom{X_1}{Y_1} \sum_{m_2 > 0} L_{m_2} \binom{X_2}{Y_2} \\
 &= \left(\sum_{m_1 > m_2 > 0} + \sum_{m_2 > m_1 > 0} + \sum_{m_1 = m_2 > 0} \right) L_{m_1} \binom{X_1}{Y_1} L_{m_2} \binom{X_2}{Y_2} \\
 &= \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1} + \sum_{m > 0} L_m \binom{X_1}{Y_1} L_m \binom{X_2}{Y_2}.
 \end{aligned}$$

Lemma

For all $m \geq 1$ we have

$$L_m \binom{X_1}{Y_1} L_m \binom{X_2}{Y_2} = \frac{L_m \binom{X_1}{Y_1+Y_2} - L_m \binom{X_2}{Y_1+Y_2}}{X_1 - X_2} + L_m \binom{X_1, X_2}{Y_1, Y_2} + L_m \binom{X_2, X_1}{Y_2, Y_1}$$

where

$$L_m \binom{X_1, X_2}{Y_1, Y_2} = L_m \binom{X_1}{Y_1 + Y_2} \left(\beta \binom{X_2 - X_1}{-Y_2} - \frac{1}{2} \right) + \beta \binom{X_1 - X_2}{Y_1} L_m \binom{X_2}{Y_1 + Y_2}$$

④ Combinatorial MES - Combinatorial MES

Proposition

For all $m \geq 1$ the series

$$L_m \binom{X}{Y} = \frac{e^{X+mY} q^m}{1 - e^X q^m},$$

$$L_m \binom{X_1, X_2}{Y_1, Y_2} = L_m \binom{X_1}{Y_1 + Y_2} \left(\beta \binom{X_2 - X_1}{-Y_2} - \frac{1}{2} \right) + \beta \binom{X_1 - X_2}{Y_1} L_m \binom{X_2}{Y_1 + Y_2},$$

$$L_m \binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} = \text{explicit long formula}$$

are **symmetril**.

Remark

- The $L_m \binom{X}{Y}$ can be seen as the generating series of "bi-monotangent" function.
- The construction of $L_m \binom{X_1, \dots, X_r}{Y_1, \dots, Y_r}$ in terms of β and $L_m \binom{X}{Y}$ corresponds to "Multitangent = MZV-linear combination of monotangent".

④ Combinatorial MES - Make \mathfrak{g} symmetril

Proposition

If L_m is **symmetril** for all $m \geq 1$, then

$$\mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) = \sum_{m>0} L_m\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right),$$

$$\mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) = \sum_{m_1 > m_2 > 0} L_{m_1}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_2 \\ Y_2 \end{matrix}\right) + \sum_{m>0} L_m\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right),$$

$$\begin{aligned} \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix}\right) &= \sum_{m_1 > m_2 > m_3 > 0} L_{m_1}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_2 \\ Y_2 \end{matrix}\right) L_{m_3}\left(\begin{matrix} X_3 \\ Y_3 \end{matrix}\right) \\ &+ \sum_{m_1 > m_2 > 0} \left(L_{m_1}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_3 \\ Y_3 \end{matrix}\right) + L_{m_1}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_2, X_3 \\ Y_2, Y_3 \end{matrix}\right) \right) \\ &\quad + \sum_{m>0} L_m\left(\begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix}\right), \end{aligned}$$

are also **symmetril**.

④ Combinatorial MES - Combinatorial MES

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

The following series are **symmetril** and **swap invariant**

$$\mathfrak{G}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) = \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) + \beta\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right),$$

$$\mathfrak{G}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) = \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) + \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_2 \\ Y_2 \end{matrix}\right)\beta\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) + \beta\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right),$$

$$\begin{aligned} \mathfrak{G}\left(\begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix}\right) &= \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix}\right) + \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right)\beta\left(\begin{matrix} X_3 \\ Y_3 \end{matrix}\right) \\ &\quad + \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right)\beta\left(\begin{matrix} X_2, X_3 \\ Y_2, Y_3 \end{matrix}\right) + \beta\left(\begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix}\right). \end{aligned}$$

In the mould language: \mathfrak{G} is the mould product of the two symmetril bimoulds \mathfrak{g}^{il} and β .

④ Combinatorial MES - Combinatorial MES explicit

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

The following series are **symmetril** and **swap invariant**

$$\mathfrak{G}\binom{X_1}{Y_1} = \beta\binom{X_1}{Y_1} + \mathfrak{g}\binom{X_1}{Y_1},$$

$$\begin{aligned}\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} &= \beta\binom{X_1, X_2}{Y_1, Y_2} - \beta\binom{X_1 - X_2}{Y_2} \mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \frac{1}{2} \mathfrak{g}\binom{X_1}{Y_1 + Y_2} \\ &\quad + \beta\binom{X_2}{Y_2} \mathfrak{g}\binom{X_1}{Y_1} + \beta\binom{X_1 - X_2}{Y_1} \mathfrak{g}\binom{X_2}{Y_1 + Y_2} + \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2}.\end{aligned}$$

In the mould language: \mathfrak{G} is the mould product of the two symmetril bimoulds \mathfrak{g}^{il} and β .

④ Combinatorial MES - Combinatorial MES

Definition

We define the **combinatorial multiple Eisenstein series** G in depth ≤ 2 by

$$\mathfrak{G}\left(\frac{X}{Y}\right) =: \sum_{\substack{k \geq 1 \\ d \geq 0}} G\binom{k}{d} X^{k-1} \frac{Y^d}{d!},$$

$$\mathfrak{G}\left(\frac{X_1, X_2}{Y_1, Y_2}\right) =: \sum_{\substack{k_1, k_2 \geq 1 \\ d_1, d_2 \geq 0}} G\binom{k_1, k_2}{d_1, d_2} X_1^{k_1-1} X_2^{k_2-1} \frac{Y_1^{d_1}}{d_1!} \frac{Y_2^{d_2}}{d_2!}.$$

The symmetrility \mathfrak{G} of gives

$$G\binom{k_1}{d_1} G\binom{k_2}{d_2} = G\binom{k_1, k_2}{d_1, d_2} + G\binom{k_2, k_1}{d_2, d_1} + G\binom{k_1 + k_2}{d_1 + d_2}.$$

④ Combinatorial MES - The space of CMES

Definition

Space of double combinatorial multiple Eisenstein series of weight $K \geq 1$:

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1, k_2}{d_1, d_2} \mid \begin{array}{l} k+d=k_1+k_2+d_1+d_2=K \\ k, k_1, k_2 \geq 1, d, d_1, d_2 \geq 0 \end{array} \right\rangle_{\mathbb{Q}}$$

Proposition

$$\begin{aligned} q \frac{d}{dq} \mathfrak{G}\binom{X_1}{Y_1} &= \frac{d}{dX_1} \frac{d}{dY_1} \mathfrak{G}\binom{X_1}{Y_1}, \\ q \frac{d}{dq} \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} &= \left(\frac{d}{dX_1} \frac{d}{dY_1} + \frac{d}{dX_2} \frac{d}{dY_2} \right) \mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2}. \end{aligned}$$

Corollary

Combinatorial multiple Eisenstein series are closed under $q \frac{d}{dq}$. In particular

$$q \frac{d}{dq} \mathfrak{D}_K \subset \mathfrak{D}_{K+2}.$$

④ Combinatorial MES - The space of CMES

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1, k_2}{d_1, d_2} \mid \begin{array}{l} k+d=k_1+k_2+d_1+d_2=K \\ k, k_1, k_2 \geq 1, d, d_1, d_2 \geq 0 \end{array} \right\rangle_{\mathbb{Q}}$$

$$\mathfrak{D}_K^0 = \left\langle G\binom{k}{0}, G\binom{k_1, k_2}{0, 0} \in \mathfrak{D}_K \right\rangle_{\mathbb{Q}}$$

Proposition

- \mathfrak{D}_K contains the space of **quasi modular forms** $\mathbb{Q}[\tilde{G}_2, \tilde{G}_4, \tilde{G}_6]_K$ of weight K .
- \mathfrak{D}_K^0 contains the space of **modular forms** $\mathbb{Q}[\tilde{G}_4, \tilde{G}_6]_K$ of weight K

Osaka Namba Tully's & Abeno Harukas computer calculation give:

k	1	2	3	4	5	6	7	8
$\dim \mathfrak{D}_K = ?$	1	2	3	5	7	11	14	..
$\dim \mathfrak{D}_K^0 = ?$	1	2	3	3	4	4	5	5
# generators of \mathfrak{D}_K	1	3	7	14	25	41	63	92

④ Combinatorial MES - CMES are q -MZV & Bachmann Träumereien

Proposition

Combinatorial multiple Eisenstein series are q -analogues of multiple zeta values, i.e. for $k_1 \geq 2, k_2, \dots, k_r \geq 1$ we have ($r \leq 3$)

$$\lim_{q \rightarrow 1} (1-q)^{k_1+\dots+k_r} G\begin{pmatrix} k_1, \dots, k_r \\ 0, \dots, 0 \end{pmatrix} = \zeta(k_1, \dots, k_r).$$

Expectations:

- It should be possible to use our construction of combinatorial multiple Eisenstein series in all depths.
(Assuming that we have an explicit rational solution to double shuffle.)
- We expect an algebra homomorphism from the space of these series to the space of MZV, which corresponds to $q \rightarrow 1$ in the admissible case.
- The constant term ($q = 0$) is a rational solution for double shuffle.

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Thank you very much for your attention.