Checkerboard style Schur multiple zeta values

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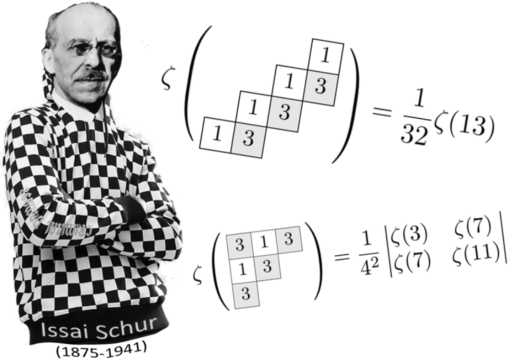
Based on joint works with Y. Yamasaki & S. Charlton



解析数論セミナー

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Overview

1

Multiple zeta values

$$\zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!}$$

2

Schur multiple zeta values

$$\zeta\left(\begin{ytableau} \none & \none & \none & h \\ \none & e & f & g \\ b & d \\ a & c \end{ytableau}\right)$$

3

Checkerboard style Schur multiple zeta values

$$\zeta\left(\begin{ytableau} \none & \none & \none & 1 \\ \none & \none & \iddots & *(lightgray) 3 \\ \none & 1 & *(lightgray) \iddots \\ 1 & *(lightgray) 3 \end{ytableau}\right) = \frac{2}{4^n}\zeta(4n+1)$$

Definition

For $k_1, \dots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \dots, k_r) = \sum_{0<m_1<\dots<m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

weight: $k_1 + \dots + k_r$, **depth**: r.

- Today we will talk about explicit evaluations of these numbers.
- In the case $r = 1$ these are just the classical Riemann zeta values

$$\zeta(k) = \sum_{m>0} \frac{1}{m^k}\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(3) \notin \mathbb{Q}\,, \qquad \zeta(4) = \frac{\pi^4}{90}\,, \dots\,.$$

① MZV - $\zeta(k, \ldots, k)$

Eulers formulas $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ are a special cases of

$$\zeta(\{2\}^n) := \zeta(\underbrace{2, ..., 2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}, \quad \zeta(\{4\}^n) = \frac{2^{2n+1}\pi^{4n}}{(4n+2)!}.$$

Both formulas can be proven easily using generating series, e.g.

$$\sum_{n=0}^{\infty} \zeta(\{2\}^n) T^{2n+1} = T \prod_{m=1}^{\infty} \left(1 + \frac{T^2}{m^2}\right) = \frac{\sin(\pi i T)}{\pi i} = \sum_{n=0}^{\infty} \frac{\pi^{2n}}{(2n+1)!} T^{2n+1}.$$

Proposition

For all $k \geq 2$ we have

$$\zeta(k, \ldots, k) \in \mathbb{Q}[\zeta(k \cdot m) \mid m \geq 1],$$

i.e. in particular $\zeta(2k, \ldots, 2k) \in \mathbb{Q}[\pi^2]$.

① MZV - $\zeta(1,3,\ldots,1,3)$

Theorem (Borwein-Bradley-Broadhurst-Lisonek)

For all $n \geq 1$ we have

$$\zeta(1,3,\ldots,1,3) = \zeta(\{1,3\}^n) = \frac{2\pi^{4n}}{(4n+2)!} = \frac{1}{4^n}\zeta(\{4\}^n)\,.$$

- This identity was first conjectured by Zagier.
- Nowadays there are various different generalization of this formula .

① MZV - $\zeta(1,3,\ldots,1,3)$ - Original proof

11.2. **Proof of Zagier's Conjecture.** Let ${}_2F_1(a,b;c;x)$ denote the Gaussian hypergeometric function. Then:

Theorem 11.1.

$$\sum_{n=0}^{\infty} L(\{3,1\}^n;x)t^{4n} \tag{11.1}$$
$$= {}_2F_1\left(\tfrac{1}{2}t(1+i),-\tfrac{1}{2}t(1+i);1;x\right){}_2F_1\left(\tfrac{1}{2}t(1-i),-\tfrac{1}{2}t(1-i);1;x\right).$$

Proof. Both sides of the putative identity start

$$1+\frac{t^4}{8}x^2+\frac{t^4}{18}x^3+\frac{t^8+44t^4}{1536}x^4+\cdots$$

and are annihilated by the differential operator

$$D_{31}:=\left((1-x)\frac{d}{dx}\right)^2\left(x\frac{d}{dx}\right)^2-t^4.$$

Once discovered, this can be checked in Mathematica or Maple. □

J. Borwein, D. Bradley, D. Broadhurst and P. Lisonek: "Special values of multiple polylogarithms", *Trans. Amer. Math. Soc.*, **353** (2001), no. 3, 907–941.

Corollary 2. (Zagier's Conjecture)[69] *For all nonnegative integers n,*

$$\zeta(\{3,1\}^n) = \frac{2\pi^{4n}}{(4n+2)!}.$$

Proof. Gauss's ${}_2F_1$ summation theorem gives

$${}_2F_1(a,-a;1;1) = \frac{1}{\Gamma(1-a)\Gamma(1+a)} = \frac{\sin(\pi a)}{\pi a}.$$

Hence, setting $x = 1$ in the generating function (11.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \zeta(\{3,1\}^n) t^{4n} \\
&= {}_2F_1\left(\tfrac{1}{2}t(1+i), -\tfrac{1}{2}t(1+i); 1; 1\right) {}_2F_1\left(\tfrac{1}{2}t(1-i), -\tfrac{1}{2}t(1-i); 1; 1\right) \\
&= \frac{2\sin(\frac{1}{2}(1+i)\pi t)\sin(\frac{1}{2}(1-i)\pi t)}{\pi^2 t^2} \\
&= \frac{\cosh(\pi t) - \cos(\pi t)}{\pi^2 t^2} \\
&= \sum_{n=0}^{\infty} \frac{2\pi^{4n} t^{4n}}{(4n+2)!}.
\end{aligned}$$

$\square$

J. Borwein, D. Bradley, D. Broadhurst and P. Lisonek: "Special values of multiple polylogarithms", *Trans. Amer. Math. Soc.*, **353** (2001), no. 3, 907–941.

① MZV - $\zeta(1,3,\ldots,1,3)$ - Combinatorial proof

The identity $\zeta(\{1,3\}^n) = \frac{1}{4^n}\zeta(\{4\}^n)$ can be proven by using (finite) double shuffle relations (Borwein-Bradley-Broadhurst):

$$\sum_{r=-n}^{n} (-1)^r \zeta(\{2\}^{n-r})\zeta(\{2\}^{n+r}) \overset{\text{shuffle product}}{=} 4^n \zeta(\{1,3\}^n)\,,$$
$$\sum_{r=-n}^{n} (-1)^r \zeta(\{2\}^{n-r})\zeta(\{2\}^{n+r}) \overset{\text{stuffle product}}{=} \zeta(\{4\}^n)\,.$$

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$$\sum_{r=-n}^{n} (-1)^r \zeta(\{2\}^{n-r})\zeta(\{2\}^{n+r}) \overset{\text{stuffle product}}{=} \zeta(\{4\}^n)\,.$$

Remark

This implies that this identity also holds for Multiple Eisenstein-series, i.e.

$$G_{\{1,3\}^n}(\tau) = \frac{1}{4^n} G_{\{4\}^n}(\tau)\,.$$

But in contrast to MZV they are both in general not multiples of $G_{4n}(\tau)$ if $n \geq 3$, because of the existence of cusp forms.

There is also a 3-1-3-Formula:

Theorem (Bowman-Bradley)

For all $n \geq 1$ we have

$$\zeta(3, \{1,3\}^n) = \frac{1}{4^n} \sum_{k=0}^{n} (-1)^k \zeta(4k+3)\zeta(\{4\}^{n-k})\,.$$

The proof is again done by guessing the correct generating series and show that it vanishes under a certain differential operator.

Definition

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta-star value** (MZSV)

$$\zeta^\star(k_1, \ldots, k_r) = \sum_{0<m_1 \leq \cdots \leq m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

Theorem (Muneta)

For all $n \geq 1$ we have

$$\zeta^\star(\{1,3\}^n) = \text{complicated but explicit coefficient} \cdot \pi^{4n} \in \mathbb{Q}\pi^{4n}\,.$$

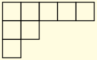
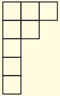
Goal

- Introduce Schur multiple zeta values as a generalization of MZV and MZSV.
- Show 13-formulas for Schur multiple zeta values.

- By a **partition** (of $\lambda_1 + \cdots + \lambda_h$) we denote a tuple $\lambda = (\lambda_1, \ldots, \lambda_h)$ with $\lambda_1 \geq \cdots \geq \lambda_h \geq 1$.
- Its **transpose** is denoted by $\lambda' = (\lambda'_1, \ldots, \lambda'_{h'})$ and it is defined by transposing the corresponding Young diagram.

Example

A partition and its transpose visualized by Young diagrams

$\lambda = (5, 2, 1) =$  $\lambda' = (3, 2, 1, 1, 1) =$ 

Let $\lambda = (\lambda_1, \ldots, \lambda_h)$ be a partition.

- For another partition $\mu = (\mu_1, \ldots, \mu_r)$ we write $\mu \subset \lambda$ if $r \leq h$ and $\mu_j < \lambda_j$ for $j = 1, \ldots, r$.
- For partitions λ, μ with $\mu \subset \lambda$ we define

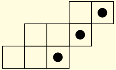
$$D(\lambda/\mu) = \left\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq h \,, \mu_i < j \leq \lambda_i \right\} \,.$$

- We denote the set of all corners of λ/μ by $\mathrm{Cor}(\lambda/\mu) \subset D(\lambda/\mu)$.

Example When $\lambda/\mu = (5,4,3)/(3,1)$ we have

$$\begin{aligned} D(\lambda/\mu) &= \{(1,4),(1,5),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3)\}\,, \\ \mathrm{Cor}(\lambda/\mu) &= \{(1,5),(2,4),(3,3)\}\,, \end{aligned}$$

which we visualize (Corners = ●) in the corresponding Young diagram:



- A (skew) Young tableau $\mathbf{k} = (k_{i,j})$ of shape λ/μ is a collection of $k_{i,j} \in \mathbb{N}$ for all $(i,j) \in D(\lambda/\mu)$.

Example When $\lambda/\mu = (5,4,3)/(3,1)$ we visualize this Young tableau by

$$\mathbf{k} \;=\; (k_{i,j}) \;=\; \begin{array}{ccccc} & & & \boxed{k_{1,4}} & \boxed{k_{1,5}} \\ & \boxed{k_{2,2}} & \boxed{k_{2,3}} & \boxed{k_{2,4}} & \\ \boxed{k_{3,1}} & \boxed{k_{3,2}} & \boxed{k_{3,3}} & & \end{array} .$$

- A Young tableau $(m_{i,j})$ is called semi-standard if $m_{i,j} < m_{i+1,j}$ and $m_{i,j} \leq m_{i,j+1}$ for all i and j.
- The set of all Young tableaux and all semi-standard Young tableaux of shape λ/μ are denoted by $T(\lambda/\mu)$ and $\mathrm{SSYT}(\lambda/\mu)$, respectively.

We call a Young tableau $\mathbf{k} = (k_{i,j}) \in T(\lambda/\mu)$ admissible if $k_{i,j} \geq 2$ for $(i,j) \in \mathrm{Cor}(\lambda/\mu)$.

Definition

For an admissible $\mathbf{k} = (k_{i,j}) \in T(\lambda/\mu)$ the **Schur multiple zeta value** is defined by

$$\zeta(\mathbf{k}) = \sum_{(m_{i,j}) \in \mathrm{SSYT}(\lambda/\mu)} \prod_{(i,j) \in D(\lambda/\mu)} \frac{1}{m_{i,j}^{k_{i,j}}}.$$

These generalize MZV and MZSV in the following way.

$$\zeta(k_1,\ldots,k_r) = \zeta\left(\begin{array}{|c|}\hline k_1 \\ \hline \vdots \\ \hline k_r \\ \hline \end{array}\right) \quad \text{and} \quad \zeta^\star(k_1,\ldots,k_r) = \zeta\left(\begin{array}{|c|c|c|}\hline k_1 & \cdots & k_r \\ \hline \end{array}\right).$$

Example 1 For $a \geq 1$ and $b, c \geq 2$ we have

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \sum_{\substack{m_a \leq m_b \\ \wedge \\ m_c}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c}\,.$$

Clearly every Schur MZV is just a linear combination of MZV, e.g.

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = \zeta(a,c,b) + \zeta(a,b,c) + \zeta(a+b,c) + \zeta(a,b+c)\,.$$

Example 2

For $a, b, d \geq 1$ and $c, e, f \geq 2$ we have

$$\zeta\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \multicolumn{1}{|c}{} \\ \cline{1-2} f & \multicolumn{1}{|c}{} & \\ \cline{1-1}\end{array}\right) = \sum_{\begin{array}{ccccc} m_a & \leq & m_b & \leq & m_c \\ \wedge & & \wedge & & \\ m_d & \leq & m_e & & \\ \wedge & & & & \\ m_f & & & & \end{array}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c \cdot m_d^d \cdot m_e^e \cdot m_f^f} .$$

Example 3

For $b, d \geq 1$ and $c, e, f \geq 2$ we have

$$\zeta\left(\begin{array}{c|c|c|}\cline{2-3} & b & c \\ \hline \multicolumn{1}{|c|}{d} & e & \multicolumn{1}{|c}{} \\ \cline{1-2} \multicolumn{1}{|c|}{f} & & \\ \cline{1-1}\end{array}\right) = \sum_{\begin{array}{ccccc} & & m_b & \leq & m_c \\ & & \wedge & & \\ m_d & \leq & m_e & & \\ \wedge & & & & \\ m_f & & & & \end{array}} \frac{1}{m_b^b \cdot m_c^c \cdot m_d^d \cdot m_e^e \cdot m_f^f} .$$

Compared to multiple zeta values, the product of two arbitrary Schur multiple zeta values can be written quite easily.

Example The harmonic product formula of MZV is given by

$$\begin{aligned}\zeta(a)\cdot\zeta(b) &= \sum_{m>0}\frac{1}{m^a}\sum_{n>0}\frac{1}{n^b} \\ &= \sum_{0<m<n}\frac{1}{m^a n^b} + \sum_{0<n<m}\frac{1}{m^a n^b} + \sum_{m=n>0}\frac{1}{m^{a+b}} \\ &= {\color{blue}\zeta(a,b)} + {\color{red}\zeta(b,a)} + {\color{blue}\zeta(a+b)}\,.\end{aligned}$$

Using the notion of Schur MZV this can be written as

$$\zeta(\boxed{a})\,\zeta(\boxed{b}) = \sum_{0<m\leq n}\frac{1}{m^a n^b} + \sum_{0<n<m}\frac{1}{m^a n^b} = {\color{blue}\zeta(\boxed{a}\boxed{b})} + {\color{red}\zeta\left(\begin{array}{|c|}\hline b\\ \hline a\\ \hline\end{array}\right)}\,.$$

Compared to multiple zeta values, the product of two arbitrary Schur multiple zeta values can be written quite easily.

Example The harmonic product formula of MZV is given by

$$\zeta(a)\cdot\zeta(b,c) = \zeta(a,b,c) + \zeta(b,a,c) + \zeta(b,c,a) + \zeta(a+b,c) + \zeta(b,a+c)\,.$$

Using the notion of Schur MZV this can be written as

$$\zeta(\,\boxed{a}\,)\,\zeta\left(\begin{array}{|c|}\hline b \\ \hline c \\ \hline\end{array}\right) = \zeta\left(\begin{array}{c|c|}\cline{2-2} & b \\ \hline \multicolumn{1}{|c|}{a} & c \\ \hline\end{array}\right) + \zeta\left(\begin{array}{|c|}\hline b \\ \hline c \\ \hline a \\ \hline\end{array}\right).$$

② Schur MZV - Products

In general the product of two Schur MZV is always the sum of two Schur MZV.

Example

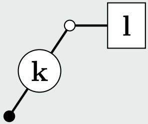
$$\zeta\left(\begin{ytableau} \none & e \\ b & d \\ a & c \end{ytableau}\right)\zeta\left(\begin{ytableau} \none & h \\ f & g \end{ytableau}\right)=\zeta\left(\begin{ytableau} \none & \none & h \\ \none & f & g \\ \none & e \\ b & d \\ a & c \end{ytableau}\right)+\zeta\left(\begin{ytableau} \none & \none & \none & h \\ \none & e & f & g \\ b & d \\ a & c \end{ytableau}\right).$$

② Schur MZV - Integral expression

Theorem (Kaneko-Yamamoto)

For every indexsets $\mathbf{k} = (k_1, \dots, k_r), \mathbf{l} = (l_1, \dots, l_s), M = k_r + l_s$ we have

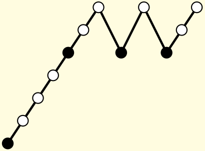
$$\zeta\left(\begin{array}{cccc} & & & k_1 \\ & & & \vdots \\ & & & k_{r-1} \\ l_1 & \dots & l_{s-1} & M \end{array}\right) =: \zeta(\mathbf{k} \circledast \mathbf{l}^\star) = I\left(\quad \right),$$



where the right-hand side is given by a Yamamoto 2-poset integral.

Example $\mathbf{k} = (4, 1), \mathbf{l} = (3, 2, 2)$:

$$\zeta\left(\begin{array}{ccc} & & 4 \\ 3 & 2 & 3 \end{array}\right) = I\left(\quad \right)$$



The result of Kaneko-Yamamoto can be generalized to arbitrary **ribbons**.

$$\begin{array}{|c|c|c|c|c|c|c|}
\cline{5-7} \multicolumn{4}{c|}{} & h & i & j \\
\cline{5-7} \multicolumn{4}{c|}{} & g & \multicolumn{2}{c}{} \\
\cline{4-5} \multicolumn{3}{c|}{} & e & f & \multicolumn{2}{c}{} \\
\hline a & b & c & d & \multicolumn{3}{c}{} \\
\cline{1-4}
\end{array}$$

Theorem (Nakasuji-Phuksuwan-Yamasaki)

Every Schur MZV of ribbon shape can be written as a Yamamoto 2-poset integral.

Open question

Can an arbitrary Schur MZV be written as a 2-poset integral?

$$\zeta\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\right) = \sum_{\substack{m_a \leq m_b \\ \wedge \quad \wedge \\ m_c \leq m_d}} \frac{1}{m_a^a \cdot m_b^b \cdot m_c^c \cdot m_d^d} = I\left(\; ? \; \right).$$

To state the Jacobi-Trudi formula we need the following notations.

- Let $T^{\mathrm{diag}}(\lambda/\mu)$ be the subset of $T(\lambda/\mu)$ consisting of all Young tableaux with the **same entries on the diagonal**.

Example

$$\begin{array}{|c|c|c|c|c|}\cline{2-5} \multicolumn{1}{c|}{} & 2 & 1 & 6 & 8 \\ \hline 9 & 5 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & 9 & 5 & 2 & \multicolumn{1}{c}{} \\ \cline{1-4} 1 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 5 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \in T^{\mathrm{diag}}((5,4,4,2,1)/(1))\,.$$

- Denote for $k_1, \dots, k_r \geq 1$ by $\zeta^*(k_1, \dots, k_r)$ the **stuffle regularized multiple zeta value** (with $\zeta^*(1) = 0$).

Example

$$\zeta^*(1) \cdot \zeta^*(2) = \zeta^*(1,2) + \zeta^*(2,1) + \zeta^*(3)\,,$$
$$\zeta^*(2,1) = -\zeta(1,2) - \zeta(3) = -2\zeta(3)\,.$$

② Schur MZV - Regularized Jacobi-Trudi formula

Let $\lambda = (\lambda_1, \dots, \lambda_h)$ and $\mu = (\mu_1, \dots, \mu_r)$ be partitions with $\mu \subset \lambda$.

Regularized Jacobi-Trudi formula (Nakasuji-Phuksuwan-Yamasaki, B.-Charlton)

For an admissible Young tableau $\mathbf{k} = (k_{i,j}) \in T^{\mathrm{diag}}(\lambda/\mu)$ and $d_{i-j} = k_{i,j}$ we have

$$\zeta(\mathbf{k}) = \det \Big(\zeta^*(d_{-\mu'_j+j-1}, d_{-\mu'_j+j-2}, \dots, d_{-\mu'_j+j-(\lambda'_i-\mu'_j-i+j)}) \Big)_{1 \le i,j \le \lambda_1} ,$$

where we set $\zeta^*(\cdots) = \begin{cases} 1 \text{ if } \lambda'_i - \mu'_j - i + j = 0 \\ 0 \text{ if } \lambda'_i - \mu'_j - i + j < 0 \end{cases}$.

Lets do the following example together on the board...

Example

$$\zeta\left(\begin{ytableau} d_0 & d_1 & d_2 \\ d_{-1} & d_0 \\ d_{-2} \end{ytableau}\right) = \begin{vmatrix} \zeta(d_{-2}, d_{-1}, d_0) & \zeta(d_{-2}, \dots, d_1) & \zeta(d_{-2}, \dots, d_2) \\ \zeta(d_0) & \zeta(d_0, d_1) & \zeta(d_0, d_1, d_2) \\ 0 & 1 & \zeta(d_2) \end{vmatrix}$$

$$\zeta\left(\begin{ytableau} \none & d_1 & d_2 \\ d_{-1} & d_0 \\ d_{-2} \end{ytableau}\right) = \begin{vmatrix} \zeta(d_{-2}, d_{-1}) & \zeta(d_{-2}, \dots, d_1) & \zeta(d_{-2}, \dots, d_2) \\ 1 & \zeta(d_0, d_1) & \zeta(d_0, d_1, d_2) \\ 0 & 1 & \zeta(d_2) \end{vmatrix}$$

- In the following we will be interested in **Checkerboard style Schur MZV**, i.e. Schur MZV with alternating entries $a \geq 1$ and $b \geq 2$.

③ Checkerboard Schur MZV - Definition

- In the following we will be interested in **Checkerboard style Schur MZV**, i.e. Schur MZV with alternating entries $a \geq 1$ and $b \geq 2$. (b is always located in the corners).
- By the Jacobi-Trudi formula these are always polynomials in the following 4-types of MZV (for some $n \geq 0$)

$$\zeta(\{a,b\}^n)\,, \quad \zeta^\star(\{b,a\}^n)\,, \quad \zeta(b,\{a,b\}^n)\,, \quad \zeta^\star(a,\{b,a\}^n)\,.$$

Example

$$\zeta\left(\begin{array}{|c|c|c|} \hline \multicolumn{2}{c|}{} & b \\ \cline{2-3} \multicolumn{1}{c|}{} & b & a \\ \cline{2-3} \multicolumn{1}{c|}{} & a & b \\ \hline a & b & \multicolumn{1}{c}{} \\ \cline{1-2} b & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}\right) = \begin{vmatrix} \zeta(a,b) & \zeta(b,\{a,b\}^2) & \zeta(b,\{a,b\}^3) \\ 1 & \zeta(b,a,b) & \zeta(b,\{a,b\}^2) \\ 0 & \zeta(b) & \zeta(b,a,b) \end{vmatrix}$$

$$= \zeta(a,b)\zeta(b,a,b)^2 + \zeta(b)\zeta(b,\{a,b\}^3)$$
$$- \zeta(b,a,b)\zeta(b,\{a,b\}^2) - \zeta(b)\zeta(a,b)\zeta(b,\{a,b\}^2)\,.$$

③ Checkerboard Schur MZV - The A and B stairs

Define for $a \geq 1, b \geq 2$ and a $n \geq 1$ and $m \geq 0$ the two Schur MZV

$$A(n) = A_{a,b}(n) = \zeta\left(\begin{array}{cccc} & & & a \\ & & \iddots & b \\ & a & \iddots & \\ a & b & & \end{array}\right), \quad B(m) = B_{a,b}(m) = \zeta\left(\begin{array}{cccc} & & a & b \\ & \iddots & b & \\ a & \iddots & & \\ b & & & \end{array}\right).$$

where n and m denote number of $\begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}$.

- It turns out that, for many shapes, the entries in the matrices in the Jacobi-Trudi formula can be written in terms of these stairs.
- In the case $(a, b) = (1, 3)$ the $A_{1,3}$ and $B_{1,3}$ have nice evaluations.

Theorem (B.-Yamasaki) (Rough statement)

The Checkerboard style Schur MZV of "thick stairs" are given by Hankel-determinants in the stairs $B(m)$.

Example

$$\zeta\left(\begin{ytableau} b & a & b & a & b \\ a & b & a & b \\ b & a & b \\ a & b \\ b \end{ytableau}\right) = \begin{vmatrix} B(0) & B(1) & B(2) \\ B(1) & B(2) & B(3) \\ B(2) & B(3) & B(4) \end{vmatrix}, \quad \zeta\left(\begin{ytableau} \none & a & b & a & b \\ a & b & a & b \\ b & a & b \\ a & b \\ b \end{ytableau}\right) = \begin{vmatrix} B(2) & B(3) \\ B(3) & B(4) \end{vmatrix},$$

$$\zeta\left(\begin{ytableau} \none & \none & b & a & b \\ \none & b & a & b \\ b & a & b \\ a & b \\ b \end{ytableau}\right) = -\begin{vmatrix} 0 & 0 & B(0) & B(1) \\ 0 & B(0) & B(1) & B(2) \\ B(0) & B(1) & B(2) & B(3) \\ B(1) & B(2) & B(3) & B(4) \end{vmatrix}, \quad \zeta\left(\begin{ytableau} \none & \none & \none & a & b \\ \none & \none & a & b \\ \none & a & b \\ a & b \\ b \end{ytableau}\right) = \begin{vmatrix} B(4) \end{vmatrix}.$$

- Also for other shapes it seems that the matrices can be written in a nice form using stairs. (current work with S. Charlton).
- Maybe there is a variation of the Jacobi-Trudi formula, which gives directly a determinant of matrices in stairs(?)

In the following we will focus on the case $(a, b) = (1, 3)$.

③ Checkerboard Schur MZV - Evaluation of $A_{1,3}$ and $B_{1,3}$

n: number of $\begin{ytableau} 1 \\ 3 \end{ytableau}$.

Theorem (B.-Yamasaki)

For any $n \geq 1$ we have

$$\zeta\left(\begin{ytableau} \none & \none & \none & 1 \\ \none & \none & \iddots & 3 \\ \none & 1 & \iddots \\ 1 & 3 \end{ytableau}\right) = \frac{2}{4^n}\zeta(4n+1), \qquad \zeta\left(\begin{ytableau} \none & \none & 1 & 3 \\ \none & \iddots & 3 \\ 1 & \iddots \\ 3 \end{ytableau}\right) = \frac{1}{4^n}\zeta(4n+3)\,.$$

n: number of $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}$.

Theorem (B.-Yamasaki)

For any $n \geq 1$ we have

$$\zeta\left(\begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{\iddots} & \boxed{3} \\ & \boxed{1} & \boxed{\iddots} & \\ \boxed{1} & \boxed{3} & & \end{array}\right) = \frac{2}{4^n}\zeta(4n+1), \qquad \zeta\left(\begin{array}{cccc} & & \boxed{1} & \boxed{3} \\ & \boxed{\iddots} & \boxed{3} & \\ \boxed{1} & \boxed{\iddots} & & \\ \boxed{3} & & & \end{array}\right) = \frac{1}{4^n}\zeta(4n+3).$$

- Similar to the original proof of the 1-3-Formula we give an explicit expression for the generating series of these numbers.
- These are also given by certain combination of hypergeometric functions.

n: number of $\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}$.

Theorem (B.-Yamasaki)

For any $n \geq 1$ we have

$$\zeta\left(\begin{array}{cccc} & & & \boxed{1} \\ & & \boxed{\iddots} & \boxed{3} \\ & \boxed{1} & \boxed{\iddots} & \\ \boxed{1} & \boxed{3} & & \end{array}\right) = \frac{2}{4^n}\zeta(4n+1), \qquad \zeta\left(\begin{array}{cccc} & & \boxed{1} & \boxed{3} \\ & \boxed{\iddots} & \boxed{3} & \\ \boxed{1} & \boxed{\iddots} & & \\ \boxed{3} & & & \end{array}\right) = \frac{1}{4^n}\zeta(4n+3)\,.$$

Example

$$\zeta\left(\begin{array}{ccc} & \boxed{1} & \boxed{3} \\ \boxed{1} & \boxed{3} & \\ \boxed{3} & & \end{array}\right) = \sum_{\begin{array}{ccc} & b_2 & \leq a_3 \\ & \wedge & \\ b_1 & \leq a_2 & \\ \wedge & & \\ a_1 & & \end{array}} \frac{1}{(a_1 a_2 a_3)^3\, b_1 b_2} = \frac{1}{16}\zeta(11)\,.$$

Question: Elementary proof for this?

First consequence of the Theorem: The thick stairs in the case $(a, b) = (1, 3)$ are Hankel-determinants in odd zeta values.

Example

$$\zeta\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1}\end{array}\right) = \frac{1}{4^2}\begin{vmatrix} \zeta(3) & \zeta(7) \\ \zeta(7) & \zeta(11) \end{vmatrix},$$

$$\zeta\left(\begin{array}{|c|c|c|c|c|}\hline 3 & 1 & 3 & 1 & 3 \\ \hline 1 & 3 & 1 & 3 & \\ \cline{1-4} 3 & 1 & 3 & & \\ \cline{1-3} 1 & 3 & & & \\ \cline{1-2} 3 & & & & \\ \cline{1-1}\end{array}\right) = \frac{1}{4^6}\begin{vmatrix} \zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(11) & \zeta(15) & \zeta(19) \end{vmatrix}.$$

③ Checkerboard Schur MZV - Thick stairs for $(a, b) = (1, 3)$

First consequence of the Theorem: The thick stairs in the case $(a, b) = (1, 3)$ are Hankel-determinants in odd zeta values.

Example

$$\zeta\left(\begin{array}{|c|c|c|c|c|} \cline{2-5} \multicolumn{1}{c|}{} & 1 & 3 & 1 & 3 \\ \hline 1 & 3 & 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} 1 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}\right) = \frac{1}{4^6} \begin{vmatrix} \zeta(11) & \zeta(15) \\ \zeta(15) & \zeta(19) \end{vmatrix},$$

$$\zeta\left(\begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & 3 & 1 & 3 \\ \cline{2-5} \multicolumn{1}{c|}{} & 3 & 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-4} 3 & 1 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} 1 & 3 & \multicolumn{3}{c}{} \\ \cline{1-2} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}\right) = -\frac{1}{4^4} \begin{vmatrix} 0 & 0 & \zeta(3) & \zeta(7) \\ 0 & \zeta(3) & \zeta(7) & \zeta(11) \\ \zeta(3) & \zeta(7) & \zeta(11) & \zeta(15) \\ \zeta(7) & \zeta(11) & \zeta(15) & \zeta(19) \end{vmatrix}.$$

Second consequence of the Theorem: We get 1-3-Formulas for the (non-admissible) stuffle regularized MZV:

Theorem (B.-Yamasaki, B.-Charlton)

For $n \geq 0$ we have

$$\zeta^*(\{1,3\}^n,1) = \frac{1}{2^{2n-1}} \sum_{j=1}^{n} (-1)^j \zeta(4j+1)\zeta(\{4\}^{n-j}),$$

$$\zeta^*(\{3,1\}^n) = \frac{1}{2^{2n-3}} \sum_{\substack{1\leq j\leq n-1 \\ 0\leq k\leq n-1-j}} (-1)^{j+k}\zeta(4j+1)\zeta(4k+3)\zeta(\{4\}^{n-j-1-k})$$
$$+ (-1)^n \sum_{k=0}^{n} \frac{1}{4^k} \zeta^\star(\{4\}^k)\zeta(\{4\}^{n-k}).$$

Third consequence of the Theorem: Every Checkerboard Schur MZV is a polynomial in odd single zetas and π^4.

Theorem (B.-Yamasaki, B.-Charlton)

Schur MZV with alternating entries in 1 and 3 are elements in $\mathbb{Q}[\pi^4, \zeta(3), \zeta(5), \dots]$.

- We can give explicit formulas for a lot of shapes as determinants in odd zeta values and powers of π^4.

Example

$$\zeta\left(\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 3 & 1 \\ \hline 3 & 1 & 3 \\ \hline \end{array}\right) = \frac{1}{32}\begin{vmatrix} \zeta(3) & \frac{\pi^4}{180} & \zeta(7) \\ \frac{\pi^4}{72} & \zeta(5) & \frac{17\pi^8}{90720} \\ \zeta(7) & \frac{13\pi^8}{226800} & \zeta(11) \end{vmatrix}$$

With the notion of Schur MZV the identity $\zeta(\{1,3\}^n) = \frac{1}{4^n}\zeta(\{4\}^n)$ reads

$$\zeta\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline \vdots\\\hline 1\\\hline 3\\\hline\end{array}\right) = \frac{1}{4^n}\zeta(\{4\}^n)\,.$$

③ Checkerboard Schur MZV - Another 1-3-Stairs Formula

With the notion of Schur MZV the identity $\zeta(\{1,3\}^n) = \frac{1}{4^n}\zeta(\{4\}^n)$ reads

$$\zeta\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline \vdots\\\hline 1\\\hline 3\\\hline\end{array}\right) = \frac{1}{4^n}\zeta(\{4\}^n)\,.$$

Theorem (B.-Yamasaki)

For any $n \geq 1$ we have

$$\zeta\left(\begin{array}{ccc}\cline{3-3} & & \multicolumn{1}{|c|}{1}\\ \cline{2-3} & \multicolumn{1}{|c|}{\cdot^{\cdot^{\cdot}}} & \multicolumn{1}{c|}{3}\\ \cline{1-3} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{\cdot^{\cdot^{\cdot}}} & \\ \cline{1-2} \multicolumn{1}{|c|}{3} & & \\ \cline{1-1}\end{array}\right) = \frac{1}{4^n}\zeta^\star(\{4\}^n)\,,$$

$$\zeta\left(\begin{array}{cccc}\cline{3-4} & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3}\\ \cline{2-4} & \multicolumn{1}{|c|}{\cdot^{\cdot^{\cdot}}} & \multicolumn{1}{c|}{\cdot^{\cdot^{\cdot}}} & \\ \cline{1-3} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & & \\ \cline{1-2}\end{array}\right) = \sum_{k=0}^{n}\frac{1}{4^k}\zeta^\star(\{4\}^k)\zeta(\{4\}^{n-k})\,,$$

where n is the number of $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3\\\hline\end{array}$ respectively.

③ Checkerboard Schur MZV - 1-2-Stairs

We have

$$A_{1,2}(n) = \zeta\left(\begin{array}{cccc} & & & 1 \\ & & \iddots & 2 \\ & 1 & \iddots & \\ 1 & 2 & & \end{array}\right) = 3\zeta(3n+1)$$

but in general it is

$$B_{1,2}(n) = \zeta\left(\begin{array}{cccc} & & 1 & 2 \\ & \iddots & 2 & \\ 1 & \iddots & & \\ 2 & & & \end{array}\right) \notin \mathbb{Q}[\zeta(k) \mid k \geq 2]\,.$$

Also easy to check:

$$\zeta\left(\begin{array}{ccc} & & 1 \\ & \iddots & 2 \\ 1 & \iddots & \\ 2 & & \end{array}\right) = \zeta^{\star}(\{3\}^n)\,.$$

Summary

- Schur MZV generalize MZV and MZSV into one object.
- The algebraic structure of Schur MZV is easy to describe.
- Checkerboard style Schur multiple zeta values can be written as matrices in stairs.
- In the 1-3-case, these stairs are odd single zeta values.
- For other values of (a, b), besides $(1, 2)$ and $(1, 3)$, there are no known results.
- There are various further open problems regarding Schur MZV.
  (work in progress with Yamasaki, Suzuki and Kadota: Sum formulas).

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- There are various further open problems regarding Schur MZV.
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ありがとうございます。