Formal quasimodular forms and formal multiple Eisenstein series

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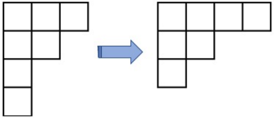


j.w. J.W. van Ittersum & Nils Matthes (in progress), Annika Burmester (arXiv:2203.09165)

Modular forms in number theory and beyond, Bielefeld University, 26th August 2022

Slides can be found here: **www.henrikbachmann.com**

Plan of this talk

Formal multiple Eisenstein series

$$\mathrm{G}^{\mathfrak{f}}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} \qquad \mathcal{G}^{\mathfrak{f}}$$

$$\cup$$

Formal quasimodular forms

$$\mathrm{G}^{\mathfrak{f}}\binom{k}{d} \qquad \widetilde{\mathcal{M}}^{\mathfrak{f}}$$

$$\cong$$

quasimodular forms

$$\widetilde{\mathcal{M}} = \mathbb{Q}[\mathrm{G}(2),\mathrm{G}(4),\mathrm{G}(6)]$$

Plan of this talk

Formal multiple Eisenstein series 2

$$\mathrm{G}^{\mathfrak{f}}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r} \qquad \mathcal{G}^{\mathfrak{f}}$$

$$\cup$$

Formal quasimodular forms 3

$$\mathrm{G}^{\mathfrak{f}}\binom{k}{d} \qquad \widetilde{\mathcal{M}}^{\mathfrak{f}}$$

$$\cong$$

quasimodular forms 1

$$\widetilde{\mathcal{M}} = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}(4), \mathrm{G}(6)]$$

4 **Multiple zeta values**

5 **Multiple Eisenstein series**


?

For $k \geq 2$ the **Eisenstein series** are defined by

$$\mathrm{G}(k) = -\frac{B_k}{2k!} + \frac{1}{(k-1)!} \sum_{n>0} \frac{n^{k-1} q^n}{1-q^n}$$

$$\overset{k \text{ even}}{=} \frac{1}{2(2\pi i)^k} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} \qquad (q = e^{2\pi i \tau}) \, .$$

The spaces of **modular forms** and **quasimodular forms** (with rational coefficients) are given by

$$\mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k = \mathbb{Q}[\mathrm{G}(4), \mathrm{G}(6)] \quad \subset \quad \widetilde{\mathcal{M}} = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}(4), \mathrm{G}(6)] \, .$$

By $\mathcal{S}_k \subset \mathcal{M}_k$ we denote the space of **cusp forms** of weight k.

Recursive formulas for Eisenstein series

- For even $k \geq 4$ we have

$$\frac{k+1}{2}\,\mathrm{G}(k) = (k-2)q\frac{d}{dq}\,\mathrm{G}(k-2) + \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 2 \text{ even}}} \mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

- For even $k \geq 6$ we have

$$\frac{(k+1)(k-1)(k-6)}{12}\,\mathrm{G}(k) = \sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 4 \text{ even}}} (k_1-1)(k_2-1)\;\mathrm{G}(k_1)\,\mathrm{G}(k_2)\,.$$

Example

$$q\frac{d}{dq}\,\mathrm{G}(2) = 5\,\mathrm{G}(4) - 2\,\mathrm{G}(2)^2, \quad \mathrm{G}(8) = \frac{6}{7}\,\mathrm{G}(4)^2, \quad \mathrm{G}(10) = \frac{10}{11}\,\mathrm{G}(4)\,\mathrm{G}(6)\,.$$

Fact: Every quasimodular form can be written as a polynomial in $\mathrm{G}(2)$ with coefficients in $\mathcal{M}$.

Derivations & $\mathfrak{sl}_2$-action

On the space $\widetilde{\mathcal{M}}$ we have the following three derivations:

- $\partial = q\frac{d}{dq}$
- $\mathfrak{d}$: Derivative with respect to $\mathrm{G}(2)$
- W: weight operator

The triple $(\partial, W, \mathfrak{d})$ satisfies the commutation relations of an $\mathfrak{sl}_2$-triple, i.e.

$$[W, \partial] = 2\partial, \quad [W, \mathfrak{d}] = -2\mathfrak{d}, \qquad [\mathfrak{d}, \partial] = W.$$

- We have $\mathcal{M} = \ker \mathfrak{d}$.
- Consequences: Rankin-Cohen brackets.

Formal multiple Eisenstein series

$$\mathrm{G}^{\mathfrak{f}}\binom{k}{d} \Longleftrightarrow \left(q\frac{d}{dq}\right)^{d} \mathrm{G}(k-d)$$

Elements

Formal symbols of the form

$$\mathrm{G}^{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$$

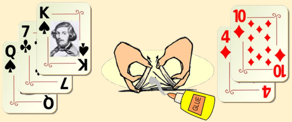
depth: $r \geq 1$

$k_1,\dots,k_r \geq 1$

$d_1,\dots,d_r \geq 0$

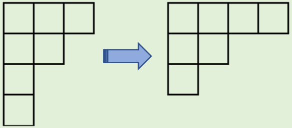
Product

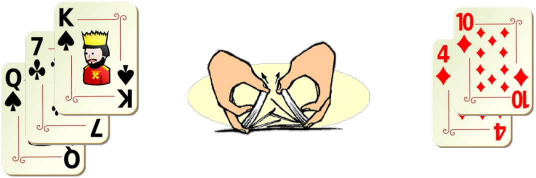
Stuffle product



Relations

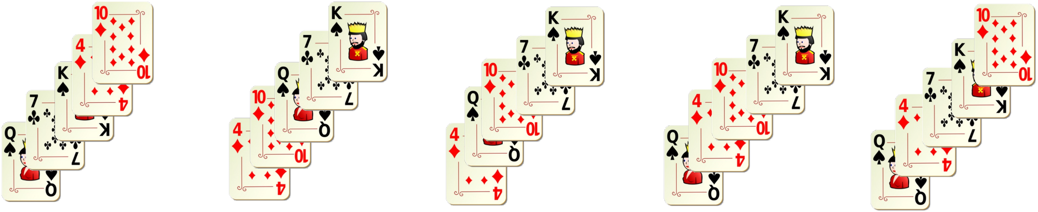
Swap invariant generating series



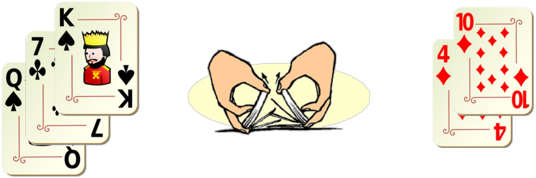


Possible shuffles of the two decks

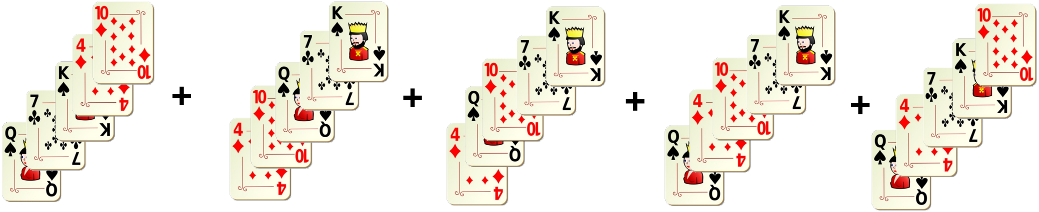
(the order of cards in the same deck stays the same)



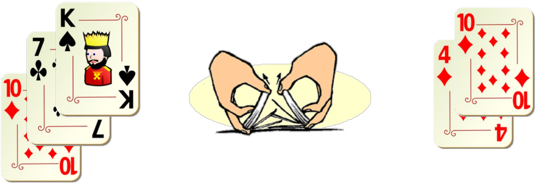
+ 5 more possibilities



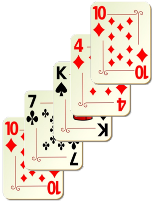
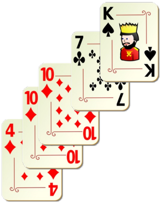
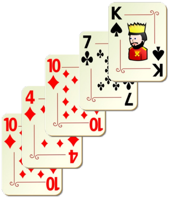
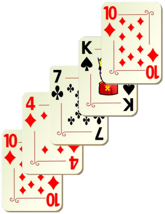
Shuffle product of two decks of cards



+ 5 more summands

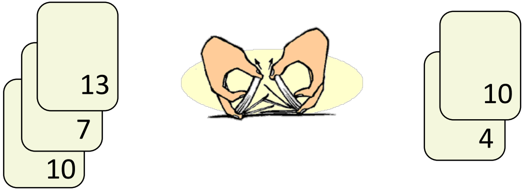


Shuffle product of two decks of cards

 + 2  +  + 

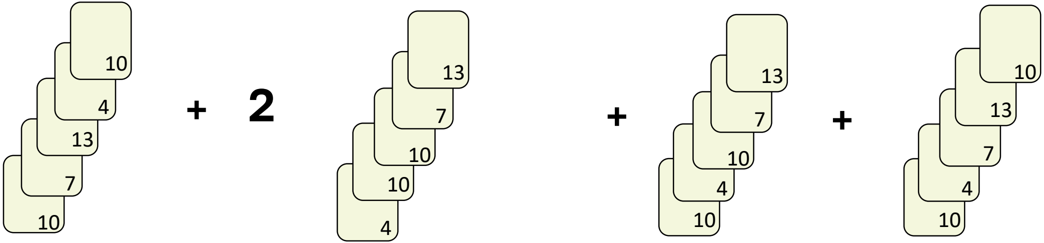
+ 5 more summands

(13,7,10)



(10,4)

Shuffle product of two tuples of numbers (decks of cards with numbers)



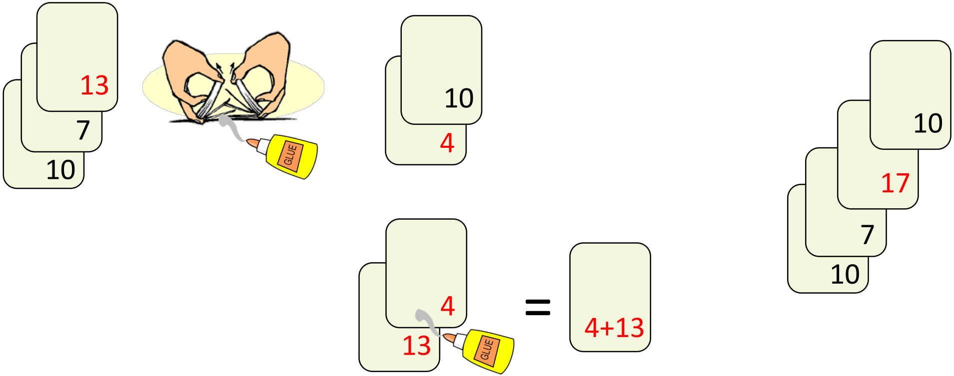
(10,4,13,7,10). + 2 (13,7,10,10,4) + (13,7,10,4,10) + (10,13,7,4,10) + …

+ 5 more summands

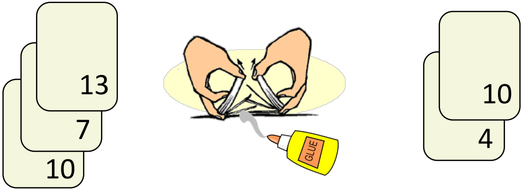
Stuffle = Shuffle + to stuff

Allow two cards from **different** decks to get stuffed together to one card

A possible term in the stuffle of these two decks

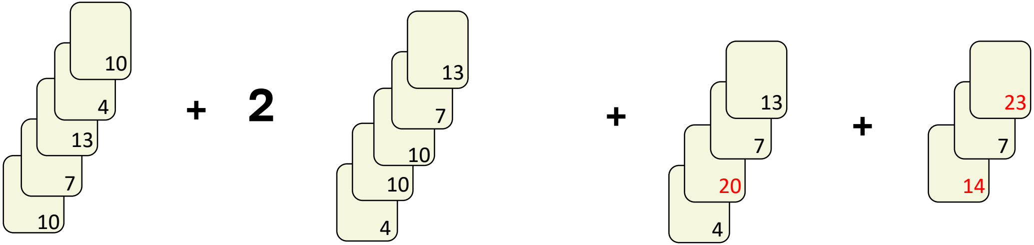


(13,7,10)



(10,4)

Stuffle product of two tuples of numbers

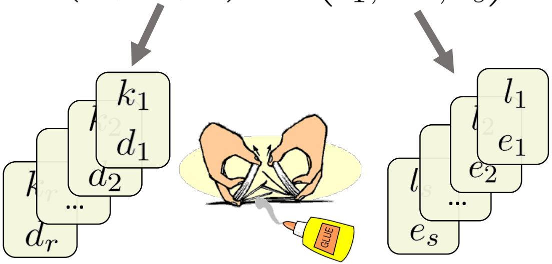


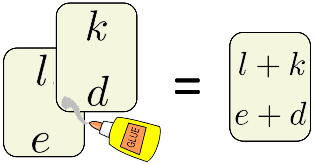
(2,6,4) + 2 (13,7,10,4,10) + (13,7,20,4) + (23, 7, 14)

+ 21 more summands

Stuffle product for formal multiple Eisenstein series

$$\mathrm{G}^{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}\;\mathrm{G}^{\mathfrak{f}}\binom{l_1,\dots,l_s}{e_1,\dots,e_s}$$





Example:

$$\mathrm{G}^{\mathfrak{f}}\binom{k_1}{d_1}\mathrm{G}^{\mathfrak{f}}\binom{k_2}{d_2} = \mathrm{G}^{\mathfrak{f}}\binom{k_1,k_2}{d_1,d_2} + \mathrm{G}^{\mathfrak{f}}\binom{k_2,k_1}{d_2,d_1} + \mathrm{G}^{\mathfrak{f}}\binom{k_1+k_2}{d_1+d_2}$$

Formal multiple Eisenstein series

$$\mathrm{G}^{\mathfrak{f}}\binom{k}{d} \Longleftrightarrow \left(q\frac{d}{dq}\right)^{d}\mathrm{G}(k-d)$$

Elements

Formal symbols of the form

$$\mathrm{G}^{\mathfrak{f}}\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix}$$

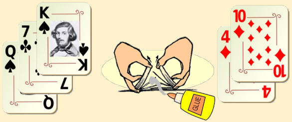
depth: $r \geq 1$

$$k_1, \dots, k_r \geq 1$$

$$d_1, \dots, d_r \geq 0$$

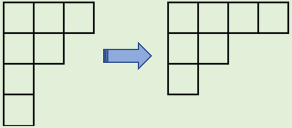
Product

Stuffle product



Relations

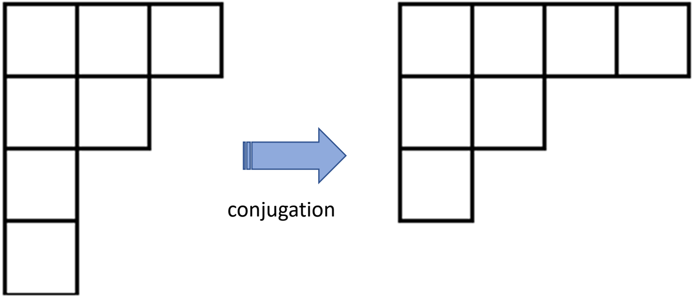
Swap invariant generating series



Conjugation of partitions

$$7 \quad = \quad 3+2+1+1 \quad = \quad 4+2+1$$

Young diagram



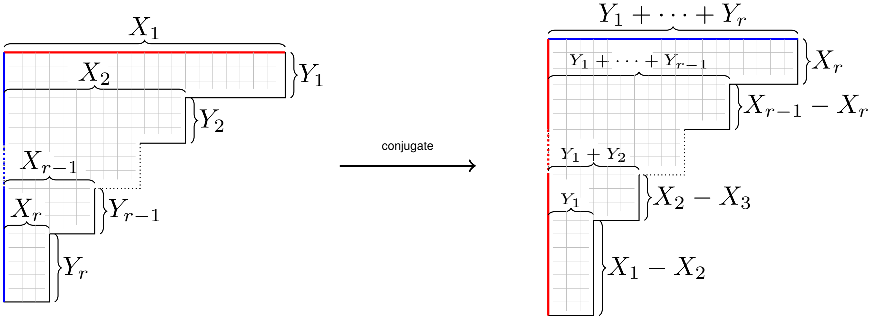
Stanley's coordinates

parts

multiplicity

$$\begin{pmatrix} 3, 2, 1 \\ 1, 1, 2 \end{pmatrix} \qquad \begin{pmatrix} 4, 2, 1 \\ 1, 1, 1 \end{pmatrix}$$

The conjugation of a Young diagram with $X_1Y_1 + \cdots + X_rY_r$ boxes and r stairs:



Conjugation on Stanley's coordinates

$$\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} \longmapsto \begin{pmatrix} Y_1 + \cdots + Y_r, \dots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \dots, X_1 - X_2 \end{pmatrix}$$

We write for the generating series of the formal multiple Eisensteins series in depth r

$$\mathfrak{G}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} := \sum_{\substack{k_1,\dots,k_r\geq 1\\ d_1,\dots,d_r\geq 0}} \mathrm{G}^{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r} X_1^{k_1-1}\dots X_r^{k_r-1}\frac{Y_1^{d_1}}{d_1!}\dots\frac{Y_r^{d_r}}{d_r!}\,.$$

Definition (Rough version)

The **algebra of formal multiple Eisenstein series** $\mathcal{G}^{\mathfrak{f}}$ is given by the $\mathbb{Q}$-vector space spanned by the symbols $\mathrm{G}^{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$ for $r\geq 1$ equipped with the **stuffle product**, modulo the relations coming from the **swap invariance**:

$$\mathfrak{G}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \mathfrak{G}\binom{Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2}\,.$$

Claim

The elements $\mathrm{G}^{\mathfrak{f}}\binom{k}{d}$ satisfy the same algebraic relations as $\left(q\frac{d}{dq}\right)^d\mathrm{G}(k-d)$.

Now everything a little bit more precise...

Define the alphabet A by (the cards)

$$A = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \mid k \geq 1,\, d \geq 0 \right\} .$$

On $\mathbb{Q}A$ we define the product $\diamond$ for $k_1, k_2 \geq 1$ and $d_1, d_2 \geq 0$ by (the gluing)

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \diamond \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ d_1 + d_2 \end{bmatrix} .$$

This gives a commutative non-unital $\mathbb{Q}$-algebra $(\mathbb{Q}A, \diamond)$.

$\mathbb{Q}\langle A\rangle$: non-commutative polynomial ring in $A = \left\{ \begin{bmatrix} k \\ d \end{bmatrix} \mid k \geq 1,\ d \geq 0 \right\}$. ($\mathbb{Q}$-lin. comb. of decks of cards)

Definition

Define the **stuffle product** $*$ on $\mathbb{Q}\langle A\rangle$ as the $\mathbb{Q}$-bilinear product, which satisfies $1 * w = w * 1 = w$ for any word $w \in \mathbb{Q}\langle A\rangle$ and

$$aw * bv = a(w * bv) + b(aw * v) + (a \diamond b)(w * v)$$

for any letters $a, b \in A$ and words $w, v \in \mathbb{Q}\langle A\rangle$.

Proposition

$(\mathbb{Q}\langle A\rangle, *)$ is a commutative $\mathbb{Q}$-algebra.

- For $k_1,\dots,k_r \geq 1, d_1,\dots,d_r \geq 0$ we use the following notation to write words in $\mathbb{Q}\langle A\rangle$:

$$\begin{bmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{bmatrix} := \begin{bmatrix} k_1 \\ d_1 \end{bmatrix} \dots \begin{bmatrix} k_r \\ d_r \end{bmatrix}.$$

- **weight:** $k_1+\dots+k_r+d_1+\dots+d_r$
- **depths:** r

In smallest depths the quasi-shuffle product is given by

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} k_1, k_2 \\ d_1, d_2 \end{bmatrix} + \begin{bmatrix} k_2, k_1 \\ d_2, d_1 \end{bmatrix} + \begin{bmatrix} k_1+k_2 \\ d_1+d_2 \end{bmatrix},$$

$$\begin{bmatrix} k_1 \\ d_1 \end{bmatrix} * \begin{bmatrix} k_2, k_3 \\ d_2, d_3 \end{bmatrix} = \begin{bmatrix} k_1, k_2, k_3 \\ d_1, d_2, d_3 \end{bmatrix} + \begin{bmatrix} k_2, k_1, k_3 \\ d_2, d_1, d_3 \end{bmatrix} + \begin{bmatrix} k_2, k_3, k_1 \\ d_2, d_3, d_1 \end{bmatrix} + \begin{bmatrix} k_1+k_2, k_3 \\ d_1+d_2, d_3 \end{bmatrix} + \begin{bmatrix} k_1, k_2+k_3 \\ d_1, d_2+d_3 \end{bmatrix}.$$

We define in depth $r \geq 1$ by the following formal power series in $\mathbb{Q}\langle A\rangle[[X_1, Y_1, \ldots, X_r, Y_r]]$

$$\mathfrak{A}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} := \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} \begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix} X_1^{k_1-1} \ldots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \ldots \frac{Y_r^{d_r}}{d_r!} .$$

With this the quasi-shuffle product in smallest depths reads

$$\mathfrak{A}\binom{X_1}{Y_1} * \mathfrak{A}\binom{X_2}{Y_2} = \mathfrak{A}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{A}\binom{X_2, X_1}{Y_2, Y_1} + \frac{\mathfrak{A}\binom{X_1}{Y_1+Y_2} - \mathfrak{A}\binom{X_2}{Y_1+Y_2}}{X_1 - X_2} .$$

② Formal MES - Swap = Conjugation of the variables in the gen. series

Definition

We define the **swap** as the linear map $\sigma : \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A\rangle$ by $\sigma(1) = 1$ and for $r \geq 1$ on the generators of $\mathbb{Q}\langle A\rangle$ by

$$\sigma\left(\mathfrak{A}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}\right) := \mathfrak{A}\binom{Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2},$$

where σ is applied coefficient-wise on the left, i.e. $\sigma\left(\begin{bmatrix}k_1,\dots,k_r\\d_1,\dots,d_r\end{bmatrix}\right)$ is defined as the coefficient of $X_1^{k_1-1}\dots X_r^{k_r-1}\frac{Y_1^{d_1}}{d_1!}\dots\frac{Y_r^{d_r}}{d_r!}$ on the right-hand side.

$$\sigma\left(\begin{bmatrix}k\\d\end{bmatrix}\right) = \frac{d!}{(k-1)!}\begin{bmatrix}d+1\\k-1\end{bmatrix}, \qquad (k \geq 1, d \geq 0).$$

Define S as the ideal in $(\mathbb{Q}\langle A\rangle, *)$ generated by all $\sigma(w) - w$ for $w \in \mathbb{Q}\langle A\rangle$, i.e.

$$S = \langle \sigma(w) - w \mid w \in \mathbb{Q}\langle A\rangle\rangle_{\mathbb{Q}} * \mathbb{Q}\langle A\rangle\,.$$

Definition

The algebra of **formal multiple Eisenstein series** is defined by

$$\mathcal{G}^{\mathfrak{f}} = {}^{\mathbb{Q}\langle A\rangle}\!\big/_{S}$$

and we denote the class of a word $\left[\begin{smallmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{smallmatrix}\right]$ by $\mathrm{G}^{\mathfrak{f}}\left(\begin{smallmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{smallmatrix}\right)$.

Example

$$\mathrm{G}^{\mathfrak{f}}\binom{k}{d} = \frac{d!}{(k-1)!}\,\mathrm{G}^{\mathfrak{f}}\binom{d+1}{k-1}, \qquad (k \geq 1, d \geq 0)\,.$$

Let $\partial : (\mathbb{Q}A, \diamond) \to (\mathbb{Q}A, \diamond)$ be the derivation defined for $k \geq 1$, $d \geq 0$ by

$$\partial\left(\begin{bmatrix} k \\ d \end{bmatrix}\right) = k \begin{bmatrix} k+1 \\ d+1 \end{bmatrix}.$$

This gives a derivation on $\mathbb{Q}\langle A\rangle$ (with respect to the concatenation product), defined by

$$\partial\left(\begin{bmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{bmatrix}\right) = \sum_{j=1}^{r} k_j \begin{bmatrix} k_1, \ldots, k_j+1, \ldots, k_r \\ d_1, \ldots, d_j+1, \ldots, d_r \end{bmatrix}.$$

Lemma

- ∂ is a derivation on $(\mathbb{Q}\langle A\rangle, *)$.
- The derivation ∂ commutes with the swap, i.e. $\partial\sigma = \sigma\partial$.

Theorem (B.-Matthes-van Ittersum 2022+)

∂ is a derivation on $(\mathcal{G}^{\mathfrak{f}}, *)$.

$$\partial\left(\mathrm{G}^{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}\right) = \sum_{j=1}^{r} k_j\,\mathrm{G}^{\mathfrak{f}}\binom{k_1,\dots,k_j+1,\dots,k_r}{d_1,\dots,d_j+1,\dots,d_r}.$$

(∂ is the formal version of $q\frac{d}{dq}$)

Conjecture

There exist a unique derivation $\mathfrak{d}$ on $(\mathbb{Q}\langle A\rangle, *)$ such that

- $\mathfrak{d}$ commutes with σ.
- The triple $(\partial, W, \mathfrak{d})$ satisfies the commutation relations of an $\mathfrak{sl}_2$-triple, i.e.

$$[W,\partial] = 2\partial, \quad [W,\mathfrak{d}] = -2\mathfrak{d}, \qquad [\mathfrak{d},\partial] = W\,,$$

where W is the weight operator, multiplying a word $\left[\begin{smallmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{smallmatrix}\right]$ by its weight $k_1+\ldots+k_r+d_1+\ldots+d_r$.

This would imply an $\mathfrak{sl}_2$-action on $\mathcal{G}^{\mathfrak{f}}$. In depth one this derivation seems to be given by

$$\mathfrak{d}\,\mathrm{G}^{\mathfrak{f}}\binom{k}{d} = d\,\mathrm{G}^{\mathfrak{f}}\binom{k-1}{d-1} - \frac{1}{2}\delta_{k+d,2}\,,$$

which correspond to the classical derivation for quasimodular forms (the derivative with respect to $\mathrm{G}(2)$).

For $k_1, \ldots, k_r \geq 1$ we write

$$\mathrm{G}^{\mathfrak{f}}(k_1, \ldots, k_r) := \mathrm{G}^{\mathfrak{f}}\binom{k_1, \ldots, k_r}{0, \ldots, 0}.$$

Instead of $*$ we will just write products of $\mathrm{G}^{\mathfrak{f}}$.

Theorem (B.-Matthes-van Ittersum 2022+)

For all $k_1, k_2 \geq 1$ with $k = k_1 + k_2 \geq 4$ even we have

$$\frac{1}{2}\left(\binom{k_1+k_2}{k_2} - (-1)^{k_1}\right)\mathrm{G}^{\mathfrak{f}}(k) = \sum_{\substack{j=2 \\ j \text{ even}}}^{k-2}\left(\binom{k-j-1}{k_1-1} + \binom{k-j-1}{k_2-1} - \delta_{j,k_1}\right)\mathrm{G}^{\mathfrak{f}}(j)\,\mathrm{G}^{\mathfrak{f}}(k-j)$$
$$+ \frac{1}{2}\left(\binom{k-3}{k_1-1} + \binom{k-3}{k_2-1} + \delta_{k_1,1} + \delta_{k_2,1}\right)\mathrm{G}^{\mathfrak{f}}\binom{k-1}{1}.$$

Corollary

- For even $k \geq 4$ we have

$$\frac{k+1}{2}\,\mathrm{G}^{\mathfrak{f}}(k) = \mathrm{G}^{\mathfrak{f}}\binom{k-1}{1} + \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 2 \text{ even}}} \mathrm{G}^{\mathfrak{f}}(k_1)\,\mathrm{G}^{\mathfrak{f}}(k_2)\,.$$

- For all even $k \geq 6$ we have

$$\frac{(k+1)(k-1)(k-6)}{12}\,\mathrm{G}^{\mathfrak{f}}(k) = \sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 4 \text{ even}}} (k_1-1)(k_2-1)\;\mathrm{G}^{\mathfrak{f}}(k_1)\,\mathrm{G}^{\mathfrak{f}}(k_2)\,.$$

Example

$$\mathrm{G}^{\mathfrak{f}}(8) = \frac{6}{7}\,\mathrm{G}^{\mathfrak{f}}(4)^2, \quad \mathrm{G}^{\mathfrak{f}}(10) = \frac{10}{11}\,\mathrm{G}^{\mathfrak{f}}(4)\,\mathrm{G}^{\mathfrak{f}}(6), \quad \mathrm{G}^{\mathfrak{f}}(12) = \frac{84}{143}\,\mathrm{G}^{\mathfrak{f}}(4)\,\mathrm{G}^{\mathfrak{f}}(8) + \frac{50}{143}\,\mathrm{G}^{\mathfrak{f}}(6)^2\,.$$

$$\widehat{\mathcal{G}}^{\mathfrak{f}} = \mathbb{Q} + \langle \mathrm{G}^{\mathfrak{f}}(k_1, \ldots, k_r) \mid r \geq 1, k_1, \ldots, k_r \geq 1 \rangle_{\mathbb{Q}} \subset \mathcal{G}^{\mathfrak{f}} .$$

By the definition of the quasi-shuffle product, it is easy to see that $(\widehat{\mathcal{G}}^{\mathfrak{f}}, *)$ is a subalgebra of $(\mathcal{G}^{\mathfrak{f}}, *)$.
Applying ∂ to the generators of $\widehat{\mathcal{G}}^{\mathfrak{f}}$ gives

$$\partial\left(\mathrm{G}^{\mathfrak{f}}(k_1, \ldots, k_r)\right) = \sum_{j=1}^{r} k_j \, \mathrm{G}^{\mathfrak{f}}\begin{pmatrix} k_1, \ldots, k_j + 1, \ldots, k_r \\ 0, \ldots, 1, \ldots, 0 \end{pmatrix} .$$

Proposition (B.-Matthes-van Ittersum 2022+)

$\widehat{\mathcal{G}}^{\mathfrak{f}}$ is closed under ∂.

Conjecture

We have $\widehat{\mathcal{G}}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}}$.

Definition

We define the algebra of **formal quasimodular forms** $\widetilde{\mathcal{M}}^{\mathfrak{f}}$ as the smallest subalgebra of $\mathcal{G}^{\mathfrak{f}}$ which satisfies the following two conditions

- $\mathrm{G}^{\mathfrak{f}}(2) \in \widetilde{\mathcal{M}}^{\mathfrak{f}}$.
- $\widetilde{\mathcal{M}}^{\mathfrak{f}}$ is closed under ∂.

Proposition (B.-Matthes-van Ittersum 2022+)

- We have $\widetilde{\mathcal{M}}^{\mathfrak{f}} = \mathbb{Q}[\mathrm{G}^{\mathfrak{f}}(2), \mathrm{G}^{\mathfrak{f}}(4), \mathrm{G}^{\mathfrak{f}}(6)] = \mathbb{Q}[\mathrm{G}^{\mathfrak{f}}(2), \partial\,\mathrm{G}^{\mathfrak{f}}(2), \partial^2\,\mathrm{G}^{\mathfrak{f}}(2)] \cong \widetilde{\mathcal{M}}$.
- The Ramanujan differential equations are satisfied:

$$\begin{aligned}\partial\,\mathrm{G}^{\mathfrak{f}}(2) &= 5\,\mathrm{G}^{\mathfrak{f}}(4) - 2\,\mathrm{G}^{\mathfrak{f}}(2)^2\,,\\ \partial\,\mathrm{G}^{\mathfrak{f}}(4) &= 8\,\mathrm{G}^{\mathfrak{f}}(6) - 14\,\mathrm{G}^{\mathfrak{f}}(2)\,\mathrm{G}^{\mathfrak{f}}(4)\,,\\ \partial\,\mathrm{G}^{\mathfrak{f}}(6) &= \frac{120}{7}\,\mathrm{G}^{\mathfrak{f}}(4)^2 - 12\,\mathrm{G}^{\mathfrak{f}}(2)\,\mathrm{G}^{\mathfrak{f}}(6)\,.\end{aligned}$$

- For $m \geq 1$ we have $\mathrm{G}^{\mathfrak{f}}(2m) \in \widetilde{\mathcal{M}}^{\mathfrak{f}}$ and

$$\mathrm{G}^{\mathfrak{f}}(2m) = -\frac{B_{2m}}{2(2m)!}(-24\,\mathrm{G}^{\mathfrak{f}}(2))^m + \text{terms with } \partial\,\mathrm{G}^{\mathfrak{f}}(2) \text{ and } \partial^2\,\mathrm{G}^{\mathfrak{f}}(2)\,.$$

- For $m \geq 2$ we have $\mathrm{G}^{\mathfrak{f}}(2m) \in \mathbb{Q}[\mathrm{G}^{\mathfrak{f}}(4), \mathrm{G}^{\mathfrak{f}}(6)]$.

- Clearly $\mathbb{Q}[\mathrm{G}^{\mathfrak{f}}(4), \mathrm{G}^{\mathfrak{f}}(6)]$ will be the space of formal modular forms.
- But how can we define the space of formal cusp forms?

Question

What are the "constant terms" of formal multiple Eisenstein series?

Philosophy: The constant terms of formal multiple Eisenstein series should behave like (formal) multiple zeta values (details later).

Idea: Divide out the relations which are satisfied by multiple zeta values.

We define the following two subsets of the alphabet A

$$A_0 = \left\{ \begin{bmatrix} k \\ 0 \end{bmatrix} \mid k \geq 1 \right\}, \qquad A^1 = \left\{ \begin{bmatrix} 1 \\ d \end{bmatrix} \mid d \geq 0 \right\}.$$

With this we define the following ideal in $(\mathbb{Q}\langle A \rangle, *)$ generated by the set $A^* \backslash (A^1)^*(A_0)^*$

$$N = \left(A^* \backslash (A^1)^*(A_0)^* \right)_{\mathbb{Q}\langle A \rangle},$$

The elements in $A^* \backslash (A^1)^*(A_0)^*$ are exactly those elements which are <u>not</u> of the form

$$\begin{bmatrix} 1, \dots, 1, k_1, \dots, k_r \\ d_1, \dots, d_s, 0, \dots, 0 \end{bmatrix}.$$

Definition

The algebra of **formal multiple zeta values** is defined by

$$\mathcal{Z}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}} \big/ N \,.$$

We denote the canonical projection by

$$\pi : \mathcal{G}^{\mathfrak{f}} \longrightarrow \mathcal{Z}^{\mathfrak{f}} \,.$$

This map can be seen as the formal version of the "projection onto the constant term".

Proposition (B.-Matthes-van Ittersum 2022+)

We have $\partial\mathcal{G}^{\mathfrak{f}} \subset \ker(\pi)$.

Claim: The ideals N captures the additional relations satisfied by multiple zeta values, which are not satisfied by multiple Eisenstein series. (More details later)

Definition

- The algebra of **formal modular forms** $\mathcal{M}^{\mathfrak{f}}$ is defined as the subalgebra of $\mathcal{G}^{\mathfrak{f}}$ generated by all $\mathrm{G}^{\mathfrak{f}}(k)$ with $k \geq 4$ even. (Conjectural definition: $\mathcal{M}^{\mathfrak{f}} = \ker \mathfrak{d}_{|\widetilde{\mathcal{M}}^{\mathfrak{f}}}$)
- We define the algebra of **formal cusp forms** by $\mathcal{S}^{\mathfrak{f}} = \ker \pi_{|\mathcal{M}^{\mathfrak{f}}}$.

The first example of a non-zero formal cusp form appears in weight 12 and we write

$$\Delta^{\mathfrak{f}} = 2400 \cdot 6! \cdot \mathrm{G}^{\mathfrak{f}}(4)^3 - 420 \cdot 7! \cdot \mathrm{G}^{\mathfrak{f}}(6)^2 \,.$$

Proposition (B.-Matthes-van Ittersum 2022+)

- We have $\mathcal{M}^{\mathfrak{f}} = \mathbb{Q}[\mathrm{G}^{\mathfrak{f}}(4), \mathrm{G}^{\mathfrak{f}}(6)] \cong \mathcal{M}$ and $\mathcal{M}^{\mathfrak{f}}_k = \mathbb{Q}\, \mathrm{G}^{\mathfrak{f}}(k) \oplus \mathcal{S}^{\mathfrak{f}}_k \cong \mathcal{M}_k$.
- We have $\Delta^{\mathfrak{f}} \in \mathcal{S}^{\mathfrak{f}}_{12}$ and $\partial \Delta^{\mathfrak{f}} = -24\, \mathrm{G}^{\mathfrak{f}}(2) \Delta^{\mathfrak{f}}$. (c.f. Claire's talk yesterday)

$$\frac{1}{432}\Delta^{\mathfrak{f}} = 48\, \mathrm{G}^{\mathfrak{f}}(2)^2 \partial\, \mathrm{G}^{\mathfrak{f}}(2)^2 + 32 \partial\, \mathrm{G}^{\mathfrak{f}}(2)^3 - 32\, \mathrm{G}^{\mathfrak{f}}(2)^3 \partial^2\, \mathrm{G}^{\mathfrak{f}}(2) - 24\, \mathrm{G}^{\mathfrak{f}}(2) \partial\, \mathrm{G}^{\mathfrak{f}}(2) \partial^2\, \mathrm{G}^{\mathfrak{f}}(2) - \partial^2\, \mathrm{G}^{\mathfrak{f}}(2)^2 \,.$$

Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight k.

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}\,.$$

There are two different ways to express the product of MZV in terms of MZV.

Stuffle product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}} \\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}} \\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j, k_1+k_2-j)\,.$$

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\zeta(2)\cdot\zeta(3) \overset{\text{stuffle}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)$$
$$\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0}\frac{1}{m^2n} = \zeta(2,1) = \zeta(3) = \sum_{m>0}\frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations

$\rightsquigarrow$ **extended double shuffle relations**.

Conjecture

All relations among MZVs are consequences of the extended double shuffle relations.

Conjecture

The space $\mathcal{Z}$ is graded by weight, i.e.

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k \, .$$

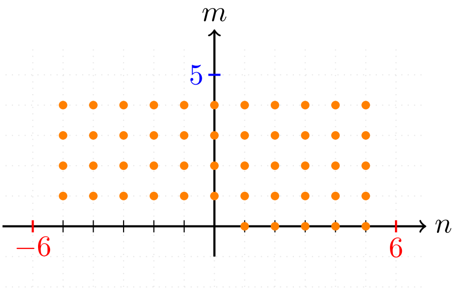
- There are various different families of relations which conjecturally give all relations among MZV.
- There are several "modular phenomena", e.g. Broadhurst-Kreimer conjecture (see bonus slides)
- Period polynomials (c.f. Isabella's talk) can be related to relations among multiple zeta values in depth two.

For $M \geq 1$ set

$$\mathbb{Z}_M = \{m \in \mathbb{Z} \mid |m| < M\}\,.$$

and for $\tau \in \mathbb{H}$ define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2)\,.$$



All the points $\lambda \in \mathbb{Z}_5 i + \mathbb{Z}_6$ satisfying $\lambda \succ 0$.

For $M \geq 1$ set

$$\mathbb{Z}_M = \{m \in \mathbb{Z} \mid |m| < M\}.$$

and for $\tau \in \mathbb{H}$ define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2).$$

Definition

For integers $k_1, \ldots, k_r \geq 1$, and $M, N \geq 1$ we define the **truncated multiple Eisenstein series** by

$$\mathbb{G}_{M,N}(k_1, \ldots, k_r) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}_M\tau + \mathbb{Z}_N}} \frac{1}{\lambda_1^{k_1} \ldots \lambda_r^{k_r}}.$$

For $k_1, \ldots, k_r \geq 2$ the **multiple Eisenstein series** are defined by

$$\mathbb{G}(k_1, \ldots, k_r) = \lim_{M \to \infty} \lim_{N \to \infty} \mathbb{G}_{M,N}(k_1, \ldots, k_r).$$

Definition

For $k_1, \dots, k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \cdots + m_r n_r} .$$

In the case $r = 1$ these are the generating series of divisor-sums $\sigma_{k-1}(n) = \sum_{d|n} n^{k-1}$

$$g(k) = \sum_{m,n>0} \frac{n^{k-1}}{(k-1)!} q^{mn} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n ,$$

and they can be viewed as *q-analogues of multiple zeta values*, since for $k_1 \geq 2, k_2, \dots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1 - q)^{k_1 + \cdots + k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r) .$$

$$\hat{g}(k_1,\dots,k_r) := (-2\pi i)^{k_1+\cdots+k_r} g(k_1,\dots,k_r) \in \mathbb{Q}[\pi i][\![q]\!]\,.$$

Theorem (Gangl-Kaneko-Zagier 2006 ($r=2$), B. 2012 ($r \geq 2$))

For $k_1,\dots,k_r \geq 2$ there exist explicit $\alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j} \in \mathbb{Z}$, such that for $q = e^{2\pi i\tau}$ we have

$$\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{\substack{0<j<r \\ l_1+\cdots+l_r=k_1+\cdots+k_r}} \alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j}\, \zeta(l_1,\dots,l_j)\hat{g}(l_{j+1},\dots,l_r) + \hat{g}(k_1,\dots,k_r)\,.$$

In particular, $\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_{k_1,\dots,k_r}(n) q^n$ for some $a_{k_1,\dots,k_r}(n) \in \mathcal{Z}[\pi i]$.

Examples

$$\mathbb{G}(k) = \zeta(k) + \hat{g}(k)\,,$$

$$\mathbb{G}(3,2) = \zeta(3,2) + 3\zeta(3)\hat{g}(2) + 2\zeta(2)\hat{g}(3) + \hat{g}(3,2)\,.$$

$$\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{n\geq 1} a_n q^n \,.$$

- $\zeta(k_1,\dots,k_r)$ is defined for $k_1 \geq 2, k_2,\dots,k_r \geq 1$.
- $\mathbb{G}(k_1,\dots,k_r)$ is just defined for $k_1,\dots,k_r \geq 2$.

Question

Is there a natural extension of $\mathbb{G}(k_1,\dots,k_r)$ for $k_1 \geq 2, k_2,\dots,k_r \geq 1$?

Answer (B.-Tasaka): Yes. Stuffle & Shuffle regularized multiple Eisenstein series (**See speedtalk of C. Turan**)

- In the construction of these regularized version, certain swap invariant q-series appear.
- Studying their algebraic structure lead to the definition of formal MES.

Formal multiple Eisenstein series

$\mathrm{G}^{\mathfrak{f}}\binom{k_1, \dots, k_r}{d_1, \dots, d_r}$ $\quad \mathcal{G}^{\mathfrak{f}}$

$\overset{?}{\cong}$

Combinatorial (bi)multiple Eisenstein series

See speedtalk of A. Burmester

$\cup$ $\quad\quad$ $\cup$

Formal quasimodular forms

$\mathrm{G}^{\mathfrak{f}}\binom{k}{d}$ $\quad \widetilde{\mathcal{M}}^{\mathfrak{f}}$

$\cong$

quasimodular forms

$\widetilde{\mathcal{M}} = \mathbb{Q}[\mathrm{G}(2), \mathrm{G}(4), \mathrm{G}(6)]$

Proposition (B.-Matthes-van Ittersum 2022+)

The map $\pi_{|\widehat{\mathcal{G}}^{\mathfrak{f}}} : \widehat{\mathcal{G}}^{\mathfrak{f}} \to \mathcal{Z}^{\mathfrak{f}}$ is surjective.

Definition

For $k_1, \ldots, k_r \geq 1$ we define the **formal multiple zeta value** $\zeta^{\mathfrak{f}}(k_1, \ldots, k_r)$ by

$$\zeta^{\mathfrak{f}}(k_1, \ldots, k_r) = \pi(\mathrm{G}^{\mathfrak{f}}(k_1, \ldots, k_r)) \,.$$

Corollary

- *(Double shuffle relations in depth two)* For $k_1, k_2 \geq 1$ we have

$$\begin{aligned}\zeta^{\mathfrak{f}}(k_1)\zeta^{\mathfrak{f}}(k_2) &= \zeta^{\mathfrak{f}}(k_1,k_2) + \zeta^{\mathfrak{f}}(k_2,k_1) + \zeta^{\mathfrak{f}}(k_1+k_2)\\ &= \sum_{l_1+l_2=k_1+k_2} \left(\binom{l_1-1}{k_1-1} + \binom{l_1-1}{k_2-1} \right) \zeta^{\mathfrak{f}}(l_1,l_2) + \delta_{k_1+k_2,2}\zeta^{\mathfrak{f}}(2)\,.\end{aligned}$$

  In particular we obtain the relation $\zeta^{\mathfrak{f}}(3) = \zeta^{\mathfrak{f}}(2,1)$ by taking $k_1 = 1, k_2 = 2$.

- *(Euler relation)* For $m \geq 1$ we have

$$\zeta^{\mathfrak{f}}(2m) = -\frac{B_{2m}}{2(2m)!}\left(-24\zeta^{\mathfrak{f}}(2)\right)^m\,.$$

Theorem (B.-Matthes-van Ittersum 2022+)

The formal multiple zeta values satisfy exaclty the extended double shuffle relations.

- Our formal multiple zeta values are isomorphic (after switching to the shuffle regularization) to the classical definition of formal multiple zeta values (Racinet).
- There is a 1:1 correspondence between objects satisfying the extended double shuffle relations and the objects satisfying the relations in $\mathcal{Z}^{\mathfrak{f}}$.

$\operatorname{gr}_r^{\mathrm{D}} \mathcal{Z}_k$: MZV of weight k and depth r modulo lower depths MZV.

Conjecture (Broadhurst-Kreimer, 1997)

The generating series of the dimensions of the weight- and depth-graded parts of multiple zeta values is given by

$$\sum_{k,r\geq 0} \dim_{\mathbb{Q}} \left(\operatorname{gr}_r^{\mathrm{D}} \mathcal{Z}_k\right) X^k Y^r = \frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4},$$

where

$$\mathsf{E}(X) = \frac{X^2}{1-X^2}, \quad \mathsf{O}(X) = \frac{X^3}{1-X^2}, \quad \mathsf{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)} = \sum_{k\geq 0} \dim \mathcal{S}_k X^k.$$

Observe that

$$\frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4}$$
$$= 1 + \left(\mathsf{E}(X)+\mathsf{O}(X)\right)Y + \left(\left(\mathsf{E}(X)+\mathsf{O}(X)\right)\mathsf{O}(X) - \mathsf{S}(X)\right)Y^2 + \cdots.$$