q-analogues of multiple zeta values and polynomial functions on partitions

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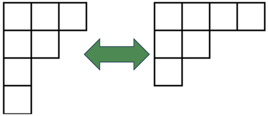
based on collaborations with: B. Brindle, A. Burmester, J.-W. van Ittersum, U. Kühn.

$$\mathbb{R} \qquad \sum_{m_1>\cdots>m_r>0} \quad \longleftrightarrow \quad \int_{1>t_1>\cdots>t_k>0}$$

$\mathbb{Q}[\![q]\!]$

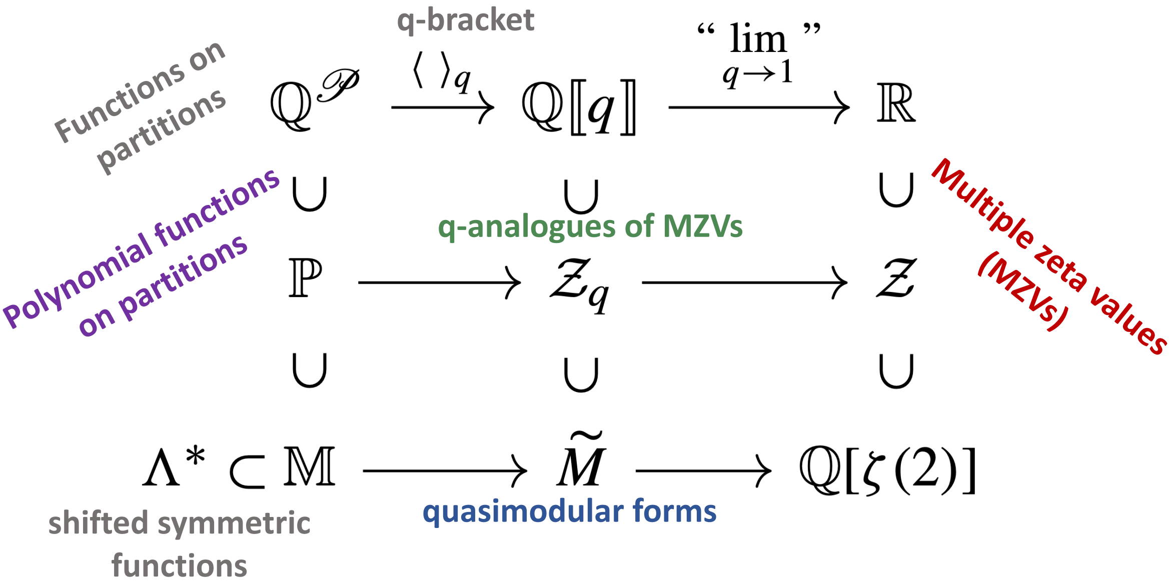


q級数とその周辺

12th & 14th March 2024

Updated slides are available here: **www.henrikbachmann.com**

The goal of my two talks will be to explain the following picture



Definition

For an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ with $k_1 \geq 2, k_2, \dots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight k.

- In the case $r = 1$ these are just the classical Riemann zeta values

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k}\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(3) \notin \mathbb{Q}\,, \qquad \zeta(4) = \frac{\pi^4}{90}\,, \quad \dots\,.$$

- MZVs were first studied by Euler ($r = 2$) and for general depth, they had their big comeback around 1990 due to their appearances in various areas of mathematics and physics.

MZVs can also be written as **iterated integrals**:

Proposition

The MZV $\zeta(k_1, ..., k_r)$ of weight $k = k_1 + ... + k_r$ can be written as an iterated integral

$$\zeta(k_1, ..., k_r) = \int_{1>t_1>\cdots>t_k>0} \omega_1(t_1)\cdots\omega_k(t_k)\,,$$

where

$$\omega_j(t) = \begin{cases} \frac{dt}{1-t} & \text{if } j \in \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_r\} \\ \frac{dt}{t} & \text{else} \end{cases}.$$

Example

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}.$$

There are two different ways to express the product of MZVs in terms of MZVs.

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \int \ldots \cdot \int \ldots = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j)\,.$$

① MZV - Double shuffle relations

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2) + 6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0} \frac{1}{m^2 n} = \zeta(2,1) = \zeta(3) = \sum_{n>0} \frac{1}{n^3}.$$

These follow from regularizing the double shuffle relations and they are called **extended double shuffle relations**.

Conjectures

- The extended double shuffle relations give all linear relations among MZV and

$$\mathcal{Z} = \bigoplus_{k\geq 0} \mathcal{Z}_k \,,$$

  i.e. there are no relations between MZV of different weight.

- (Zagier) The dimension of the spaces $\mathcal{Z}_k$ is given by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k \, X^k = \frac{1}{1 - X^2 - X^3} \,.$$

- (Hoffman) The following set gives a basis of $\mathcal{Z}$

$$\{\zeta(k_1,\dots,k_r) \mid r \geq 0, k_1,\dots,k_r \in \{2,3\}\} \,.$$

① MZV - Duality

There are other explicit families of relations among MZVs.

Definition

For any admissible index $\mathbf{k} = (k_1+1, \{1\}^{d_1-1}, \ldots, k_r+1, \{1\}^{d_r-1}) \quad (k_i,\, d_i \geq 1)$

its dual is defined as $\mathbf{k}^\vee := (d_r+1, \{1\}^{k_r-1}, \ldots, d_1+1, \{1\}^{k_1-1})$.

For example is $(2,1)^\vee = (1+1, \{1\}^{2-1})^\vee = (2+1, \{1\}^{1-1}) = (3)$.

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Theorem (Duality relation)

For every admissible index $\mathbf{k}$ *we have* $\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\vee)$.

Proofs:

- Via iterated integral representation of MZVs and the change of variables $t_i \mapsto 1 - t_i$.
- Seki-Yamamoto: Via connected sums.

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Open problem

Show that the duality relation is a consequence of the extended double shuffle relations.

$\operatorname{gr}_r \mathcal{Z}_k$: MZVs of weight k and depth r modulo lower depths MZVs.

Conjecture (Broadhurst-Kreimer, 1997)

The generating series of the dimensions of the weight- and depth-graded parts of multiple zeta values is given by

$$\sum_{k,r\geq 0} \dim_{\mathbb{Q}} (\operatorname{gr}_r \mathcal{Z}_k)\, X^k Y^r = \frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4},$$

where

$$\mathsf{E}(X) = \frac{X^2}{1-X^2}, \quad \mathsf{O}(X) = \frac{X^3}{1-X^2}, \quad \mathsf{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)} = \sum_{k\geq 0} \overbrace{\dim S_k}^{\text{cusp forms}} X^k.$$

Observe that

$$\frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4}$$
$$= 1 + \left(\mathsf{E}(X)+\mathsf{O}(X)\right)Y + \left(\left(\mathsf{E}(X)+\mathsf{O}(X)\right)\mathsf{O}(X) - \mathsf{S}(X)\right)Y^2 + \cdots.$$

M_k : Modular forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$.

Theorem (Gangl-Kaneko-Zagier, 2006)

There are at least $\dim M_k$ (linearly independent) relations among $\zeta(k)$ and the double zeta values $\zeta(a,b)$ with a, b odd and $a+b=k$. (Conjecturally these are the only ones)

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- For each Eisenstein series G_k we have $\zeta(1,k-1)+\cdots+\zeta(k-3,3)=\frac{1}{4}\zeta(k)$.
- In weight 12 we have the relation ("from" the cusp form $\Delta(q)=q\prod_{n\geq 1}(1-q^n)^{24}$)

$$28\zeta(9,3)+150\zeta(7,5)+168\zeta(5,7)=\frac{5197}{691}\zeta(12)\,.$$

Explanation: Use double shuffle relations + period polynomials or **q-analogues of MZVs**.

Rough picture

q-analogues of MZVs = q-series in $\mathbb{Q}[[q]]$ "behaving" like MZVs ($q \to 1$)

There are various motivations for studying q-analogues of MZV:

- Bridge between MZVs and modular forms. (my motivation).
- Appear in theoretical physics ($N = 4$ Super-Yang-Mills theory - Okazaki-sans talk ?).
- Connection with enumerative geometry (Hilbert schemes of points on surfaces - Yanagida-sans talk?).
- Can be used to renormalize/regularize multiple zeta values.
- Just for fun.

"Roughly speaking, in mathematics, specifically in the areas of combinatorics and special functions, a q-analogue of a theorem, identity or expression is a generalization involving a new parameter q that returns the original theorem, identity or expression in the limit as $q \to 1$. "

- The easiest example is the q-analogue of a natural number m given by

$$[m]_q = \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1}\,, \qquad \lim_{q\to 1}[m]_q = m\,.$$

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- Naive approach for q-analogue of MZV: Replace $\frac{1}{m^k}$ by $\frac{1}{[m]_q^k}$:

$$\sum_{m_1>\cdots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}} \rightsquigarrow \sum_{m_1>\cdots>m_r>0} \frac{1}{[m_1]_q^{k_1}\cdots[m_r]_q^{k_r}} \text{ `}=\text{' } \infty + O(q).$$

Problem: This sum does not make sense as an element in $\mathbb{Q}[[q]]$.

General idea

Replace $\frac{1}{m^k}$ by $\frac{P(q^m)}{[m]_q^k}$ with some good polynomial P.

In general, one can consider for polynomials $P_1 \in X\mathbb{Q}[X], P_2, \ldots, P_r \in \mathbb{Q}[X]$ the following sum

$$\sum_{m_1 > \cdots > m_r > 0} \frac{P_1(q^{m_1}) \cdots P_r(q^{m_r})}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}}.$$

These satisfy (as long as the P_i are "nice" and satisfy $P_i(1) = 1$)

$$\lim_{q \to 1} \sum_{m_1 > \cdots > m_r > 0} \frac{P_1(q^{m_1}) \cdots P_r(q^{m_r})}{[m_1]_q^{k_1} \cdots [m_r]_q^{k_r}} = \zeta(k_1, \ldots, k_r)$$

and therefore they are q-analogues of multiple zeta values.

For the connection to modular forms it is more natural to consider modified versions of q-analogues.

Definition

A **modified q-analogue of weight** k **of** $c \in \mathbb{C}$ is a q-series $f(q) \in \mathbb{C}[\![q]\!]$, such that

$$\lim_{q \to 1} (1-q)^k f(q) = c\,.$$

Proposition

Any modular form $f(q) = \sum_{n \geq 0} a_n q^n \in M_k$ is a modified q-analogue of $(2\pi i)^k a_0$ (of weight k).

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Proposition

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Modified general idea

Replace $\frac{1}{m^k}$ by $\frac{P(q^m)}{(1-q^m)^k}$ with some good polynomial P. Notice that

$$(1-q)^k \frac{P(q^m)}{(1-q^m)^k} = \frac{P(q^m)}{[m]_q^k}.$$

Definition (B.-Kühn (2017))

For $k_1, \dots, k_r \geq 1$, polynomials $P_1(X) \in X\mathbb{Q}[X]$ and $P_2(X), \dots, P_r(X) \in \mathbb{Q}[X]$ we define

$$\zeta_q(k_1, \dots, k_r; P_1, \dots, P_r) = \sum_{m_1 > \dots > m_r > 0} \frac{P_1(q^{m_1}) \cdots P_r(q^{m_r})}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}.$$

We only consider the case where $\deg(P_j) \leq k_j$ and consider the following $\mathbb{Q}$-vector space

$$\mathcal{Z}_q := \mathbb{Q} + \left\langle \zeta_q(k_1, \dots, k_r; P_1, \dots, P_r) \mid r \geq 1,\, k_1, \dots, k_r \geq 1,\, \deg(P_j) \leq k_j \right\rangle_{\mathbb{Q}}.$$

These are (modified) q-analogues of multiple zeta values:

For $k_1 \geq 2, k_2, \dots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1-q)^{k_1 + \dots + k_r} \zeta_q(k_1, \dots, k_r; P_1, \dots, P_r) = P_1(1) \cdots P_r(1)\, \zeta(k_1, \dots, k_r).$$

Similarly as for MZVs we have:

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta_q(k_1;P_1)\zeta_q(k_2;P_2) &= \sum_{m>0} \frac{P_1(q^m)}{(1-q^m)^{k_1}} \sum_{n>0} \frac{P_2(q^n)}{(1-q^n)^{k_2}} \\ &= \left(\sum_{m>n>0} + \sum_{n>m>0} + \sum_{m=n>0} \right) \frac{P_1(q^m)}{(1-q^m)^{k_1}} \frac{P_2(q^n)}{(1-q^n)^{k_2}} \\ &= \zeta_q(k_1,k_2;P_1,P_2) + \zeta_q(k_2,k_1;P_2,P_1) + \zeta_q(k_1+k_2;P_1 \cdot P_2)\,.\end{aligned}$$

In particular, $\mathcal{Z}_q$ is a $\mathbb{Q}$-algebra by using the above argument in arbitrary depth.

Recalling the results on MZV, one should have the following questions:

Questions

- What about the shuffle product?
- Iterated integrals?
- What are the relations?
- Dimension?
- Good choice of polynomials P_i?
- Why the condition $\deg(P_j) \leq k_j$?

In the following, we give an overview of several different models (choices of P_i) for q-analogues of MZVs and address some of the above questions.

One of the most classical q-analogue models was introduced by Bradley and Zhao.

Definition (Bradley (2004), Zhao (2003))

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define

$$\zeta_q^{\mathrm{BZ}}(\mathbf{k}) = \zeta_q^{\mathrm{BZ}}(k_1, \ldots, k_r) = \zeta_q(k_1, \ldots, k_r; X^{k_1-1}, \ldots, X^{k_r-1})$$
$$= \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1(k_1-1)} \cdots q^{m_r(k_r-1)}}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}.$$

Harmonic product:

$$\zeta_q^{\mathrm{BZ}}(k_1)\zeta_q^{\mathrm{BZ}}(k_2) = \zeta_q^{\mathrm{BZ}}(k_1, k_2) + \zeta_q^{\mathrm{BZ}}(k_2, k_1) + \zeta_q^{\mathrm{BZ}}(k_1 + k_2) + \zeta_q^{\mathrm{BZ}}(k_1 + k_2 - 1).$$

Results for the Bradley-Zhao model

- (Bradley 2004/Seki-Yamamoto 2019) The duality relation holds: $\zeta_q^{\mathrm{BZ}}(\mathbf{k}) = \zeta_q^{\mathrm{BZ}}(\mathbf{k}^\vee)$.
- (Takeyama 2013): Extended definition and description of an analogue of the shuffle product & some relations.

In connection with Hilbert schemes of surfaces Okounkov considers the following (See Yanagida-sans talk).

Definition (Okounkov (2014))

For $k_1, \ldots, k_r \geq 2$ define

$$\zeta_q^{\mathrm{O}}(k_1, \ldots, k_r) = \zeta_q(k_1, \ldots, k_r; O_{k_1}(X), \ldots, O_{k_r}(X)),$$

where

$$O_k(X) = \begin{cases} X^{\frac{k}{2}} & k = 2, 4, 6, \ldots \\ X^{\frac{k-1}{2}}(1+X) & k = 3, 5, 7, \ldots . \end{cases}$$

Results for the Okounkov model

- (Okounkov 2014): Dimension conjecture for the space spanned by ζ^{O} (proper subspace of $\mathcal{Z}_q$) and conjectural connections to problems in enumerative geometry.

Definition (Schlesinger (2001), Zudilin (2003), Singer (2015))

For $k_1 \geq 1, k_2, \ldots, k_r \geq 0$ define

$$\zeta_q^{\mathrm{SZ}}(\mathbf{k}) = \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r) = \zeta_q(k_1, \ldots, k_r; X^{k_1}, \ldots, X^{k_r})$$
$$= \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1 k_1} \cdots q^{m_r k_r}}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}.$$

Harmonic product (same as for MZVs):

$$\zeta_q^{\mathrm{SZ}}(k_1)\zeta_q^{\mathrm{SZ}}(k_2) = \zeta_q^{\mathrm{SZ}}(k_1, k_2) + \zeta_q^{\mathrm{SZ}}(k_2, k_1) + \zeta_q^{\mathrm{SZ}}(k_1 + k_2).$$

Results for the Schlesinger-Zudilin model

- (Zhao 2014, Ebrahimi-Fard - Manchon - Singer 2016): SZ-Duality and description of the shuffle product.
- (B.-Kühn 2017) The ζ_q^{SZ} span the space $\mathcal{Z}_q$.

Definition

- We call an index $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r$ with $k_1 \geq 1, k_2, \ldots, k_r \geq 0$ **SZ-admissible**.
- Its **weight** is given by $k_1 + \cdots + k_r + \#\{j \mid k_j = 0\}$.
- Write $\mathbf{k} = (l_1, \{0\}^{d_1}, \ldots, l_s, \{0\}^{d_s})$ $(l_i \geq 1, d_j \geq 0)$ and define its **SZ-dual** by

$$\mathbf{k}^\dagger := (d_s + 1, \{0\}^{l_s - 1}, \ldots, d_1 + 1, \{0\}^{l_1 - 1}).$$

Theorem (Zhao (2014), Singer (2014))

For every SZ-admissible index $\mathbf{k}$ *we have:* $\zeta_q^{SZ}(\mathbf{k}) = \zeta_q^{SZ}(\mathbf{k}^\dagger)$.

Example $\zeta_q^{\mathrm{SZ}}(2) = \zeta_q^{\mathrm{SZ}}(1, 0)$.

We give a combinatorial explanation of the SZ-duality later using partitions.

"Proposition"

SZ-duality + harmonic product + SZ-duality = shuffle product

Example

$$\begin{aligned}\zeta_q^{\mathrm{SZ}}(2)\zeta_q^{\mathrm{SZ}}(3) &= \zeta_q^{\mathrm{SZ}}(1,0)\zeta_q^{\mathrm{SZ}}(1,0,0)\\ &= \zeta_q^{\mathrm{SZ}}(1,0,0,1,0) + 3\zeta_q^{\mathrm{SZ}}(1,0,1,0,0) + 6\zeta_q^{\mathrm{SZ}}(1,1,0,0,0)\\ &\quad + 7\zeta_q^{\mathrm{SZ}}(1,1,0,0) + 2\zeta_q^{\mathrm{SZ}}(2,0,0) + 3\zeta_q^{\mathrm{SZ}}(2,0,0,0)\\ &\quad + \zeta_q^{\mathrm{SZ}}(1,1,0) + 2\zeta_q^{\mathrm{SZ}}(1,0,1,0)\\ &= \zeta_q^{\mathrm{SZ}}(2,3) + 3\zeta_q^{\mathrm{SZ}}(3,2) + 6\zeta_q^{\mathrm{SZ}}(4,1) + \text{(Terms of weight < 5)}.\end{aligned}$$

After multiplication with $(1-q)^5$ and taking the limit $q \to 1$, we get

$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1).$$

Theorem (B.-Kühn, 2017)

- The space $\mathcal{Z}_q$ is a $\mathbb{Q}$-algebra.
- It contains the space of (quasi-)modular forms with rational coefficients.
- It is closed under the operator $q\frac{d}{dq}$.

Similar to the double shuffle relations for MZVs we can prove relations in $\mathcal{Z}_q$, e.g.

$$\begin{aligned}-\zeta_q^{\mathrm{SZ}}(6) + 6\zeta_q^{\mathrm{SZ}}(3,3) - 3\zeta_q^{\mathrm{SZ}}(4,2) &= \zeta_q^{\mathrm{SZ}}(5) - 6\zeta_q^{\mathrm{SZ}}(2,3) - 2\zeta_q^{\mathrm{SZ}}(3,2) \\ &\quad - 5\zeta_q^{\mathrm{SZ}}(2,2) - \zeta_q^{\mathrm{SZ}}(3,1) - \zeta_q^{\mathrm{SZ}}(2,1).\end{aligned}$$

These relations are between q-analouges of mixed weight.

Question

Are there weight graded q-analogues?

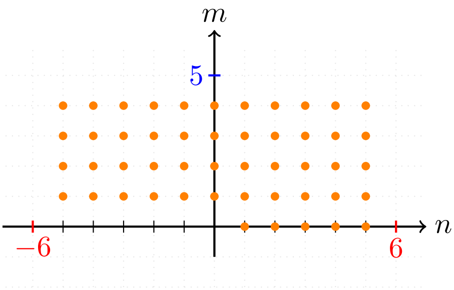
Answer: Yes! (Combinatorial) Multiple Eisenstein series.

For $M \geq 1$ set

$$\mathbb{Z}_M = \{m \in \mathbb{Z} \mid |m| < M\}\,.$$

and for $\tau \in \mathbb{H}$ define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2)\,.$$



All the points $\lambda \in \mathbb{Z}_5 i + \mathbb{Z}_6$ satisfying $\lambda \succ 0$.

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Definition

For integers $k_1, \ldots, k_r \geq 1$, and $M, N \geq 1$ we define the **truncated multiple Eisenstein series** by

$$\mathbb{G}_{M,N}(k_1, \ldots, k_r) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}_M\tau + \mathbb{Z}_N}} \frac{1}{\lambda_1^{k_1} \ldots \lambda_r^{k_r}}.$$

For $k_1, \ldots, k_r \geq 2$ the **multiple Eisenstein series** are defined by

$$\mathbb{G}(k_1, \ldots, k_r) = \lim_{M \to \infty} \lim_{N \to \infty} \mathbb{G}_{M,N}(k_1, \ldots, k_r).$$

Definition

For $k_1, \dots, k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1-1}}{(k_1-1)!} \cdots \frac{n_r^{k_r-1}}{(k_r-1)!} q^{m_1 n_1 + \dots + m_r n_r} .$$

In the case $r = 1$ these are the generating series of divisor-sums $\sigma_{k-1}(n) = \sum_{d|n} n^{k-1}$

$$g(k) = \sum_{m,n>0} \frac{n^{k-1}}{(k-1)!} q^{mn} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n ,$$

and they can be viewed as *q-analogues of multiple zeta values*, since for $k_1 \geq 2, k_2, \dots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1-q)^{k_1 + \dots + k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r) .$$

$$\hat{g}(k_1,\dots,k_r) := (-2\pi i)^{k_1+\dots+k_r} g(k_1,\dots,k_r) \in \mathbb{Q}[\pi i][\![q]\!]\,.$$

Theorem (Gangl-Kaneko-Zagier 2006 ($r=2$), B. 2012 ($r \geq 2$))

For $k_1,\dots,k_r \geq 2$ there exist explicit $\alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j} \in \mathbb{Z}$, such that for $q = e^{2\pi i\tau}$ we have

$$\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{\substack{0<j<r \\ l_1+\dots+l_r=k_1+\dots+k_r}} \alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j}\, \zeta(l_1,\dots,l_j)\hat{g}(l_{j+1},\dots,l_r) + \hat{g}(k_1,\dots,k_r)\,.$$

In particular, $\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_{k_1,\dots,k_r}(n) q^n$ *for some* $a_{k_1,\dots,k_r}(n) \in \mathcal{Z}[\pi i]$.

Examples

$$\mathbb{G}(k) = \zeta(k) + \hat{g}(k)\,,$$

$$\mathbb{G}(3,2) = \zeta(3,2) + 3\zeta(3)\hat{g}(2) + 2\zeta(2)\hat{g}(3) + \hat{g}(3,2)\,.$$

- The coefficients of $g(k_1, \ldots, k_r)$ can be seen as "multiple divisor-sums" (B.-Kühn 2013).
- They generalize the **MacMahon's generalized sums-of-divisors** $(r \geq 1)$:

$$A_r(q) = \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1 + \cdots + m_r}}{(1-q^{m_1})^2 \cdots (1-q^{m_r})^2} = g(\underbrace{2, \ldots, 2}_{r}).$$

One consequence of the formula for the Fourier expansion of multiple Eisenstein series is the following.

Theorem (B. 2024+)

We have

$$1 + \sum_{r \geq 1} A_r(q) X^{2r} = \frac{2}{X} \arcsin\left(\frac{X}{2}\right) \exp\left(\sum_{j \geq 1} \frac{(-1)^{j-1}}{j} G_{2j}(q) \left(2 \arcsin\left(\frac{X}{2}\right)\right)^{2j}\right),$$

where $G_k(q) = -\frac{B_k}{2k!} + \frac{1}{(k-1)!} \sum_{m,n \geq 1} n^{k-1} q^{mn}$.
In particular, $A_r(q)$ are quasimodular forms (Rose-Andrews 2013).

One can show in general that $g(2k, \ldots, 2k)$ for $k \geq 1$ are quasimodular forms (of mixed weight).

The g can be written in terms of ζ_q as

$$g(k_1,\dots,k_r) = \zeta_q(k_1,\dots,k_r;E_{k_1},\dots,E_{k_r}),$$

where $E_k(X)$ are the **Eulerian polynomials** defined by $\frac{E_k(X)}{(1-X)^k} = \frac{1}{(k-1)!}\sum_{d\geq 1} d^{k-1}X^d$.

Proposition

For $k_1,k_2 \geq 1$ we have

$$g(k_1)g(k_2) = g(k_1,k_2) + g(k_2,k_1) + g(k_1+k_2) + \sum_{j=1}^{k_1+k_2-1} \lambda^j_{k_1,k_2} g(j)$$

for some explicit $\lambda^j_{k_1,k_2} \in \mathbb{Q}$.

Example $g(2)g(3) = g(2,3) + g(3,2) + g(5) - \frac{1}{12}g(3)$.

Now consider the derivative $q\frac{d}{dq}$

$$q\frac{d}{dq}g_k(q) = q\frac{d}{dq}\sum_{\substack{m>0\\n>0}} \frac{n^{k-1}}{(k-1)!}q^{mn} = \sum_{\substack{m>0\\n>0}} \frac{mn^k}{(k-1)!}q^{mn}\,.$$

We see that after taking the derivative we also have a m appearing in the numerator. Moreover if we would take the d-th derivative we would get

$$\left(q\frac{d}{dq}\right)^d g_k(q) = \sum_{\substack{m>0\\n>0}} \frac{m^d n^{k+d-1}}{(k-1)!}q^{mn}\,.$$

This leads us to define g in a more general way.

Definition

For $k_1, \dots k_r \geq 1, d_1, \dots, d_r \geq 0$ define the q-series

$$g\binom{k_1, \dots, k_r}{d_1, \dots, d_r} = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{m_1^{d_1} n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{m_r^{d_r} n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \dots + m_r n_r} .$$

We say that this has **weight** $k_1 + \dots + k_r + d_1 + \dots + d_r$.

With the same idea as before we get

$$q \frac{d}{dq} g\binom{k_1, \dots, k_r}{d_1, \dots, d_r} = \sum_{j=1}^{r} k_j g\binom{k_1, \dots .k_j + 1, \dots, k_r}{d_1, \dots, d_j + 1, \dots, d_r} .$$

What about their product?

The product for the harmonic product generalizes easily by just adding the d_j:

Proposition

For $k_1, k_2 \geq 1, d_1, d_2 \geq 0$ we have

$$g\binom{k_1}{d_1} g\binom{k_2}{d_2} = g\binom{k_1, k_2}{d_1, d_2} + g\binom{k_2, k_1}{d_2, d_1} + g\binom{k_1 + k_2}{d_1 + d_2} + \sum_{j=1}^{k_1+k_2-1} \lambda_{k_1,k_2}^{j} g\binom{j}{d_1 + d_2},$$

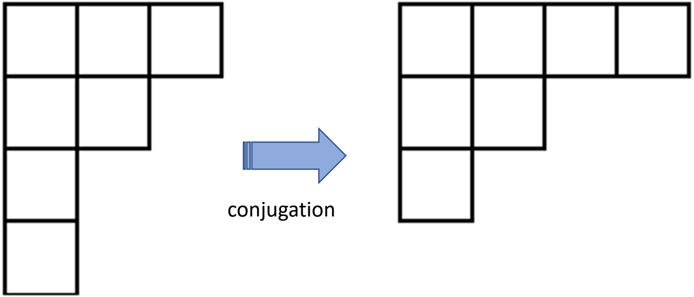
where $\lambda_{k_1,k_2}^{j} \in \mathbb{Q}$ is the same as before.

The harmonic product looks more complicated (compared to the SZ-model), but we can relate the double indexed g nicely to partitions.

Conjugation of partitions

$$7 \quad = \quad 3 + 2 + 1 + 1 \quad = \quad 4 + 2 + 1$$

Young diagram



Stanley's coordinates

parts →
multiplicity →

$$\begin{pmatrix} 3, 2, 1 \\ 1, 1, 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 4, 2, 1 \\ 1, 1, 1 \end{pmatrix}$$

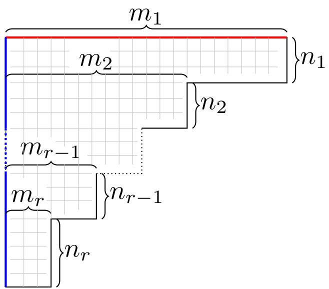
③ Multiple Eisenstein series & the q-series g - Connection with partitions

$$g\binom{k_1,\dots,k_r}{d_1,\dots,d_r} = \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r>0}} \underbrace{\frac{m_1^{d_1} n_1^{k_1-1}}{(k_1-1)!}\cdots\frac{m_r^{d_r} n_r^{k_r-1}}{(k_r-1)!}}_{f(\lambda)} q^{m_1 n_1+\dots+m_r n_r} = \sum_{N>0}\left(\sum_{\lambda\in \mathrm{Part}_r(N)} f(\lambda)\right) q^N .$$

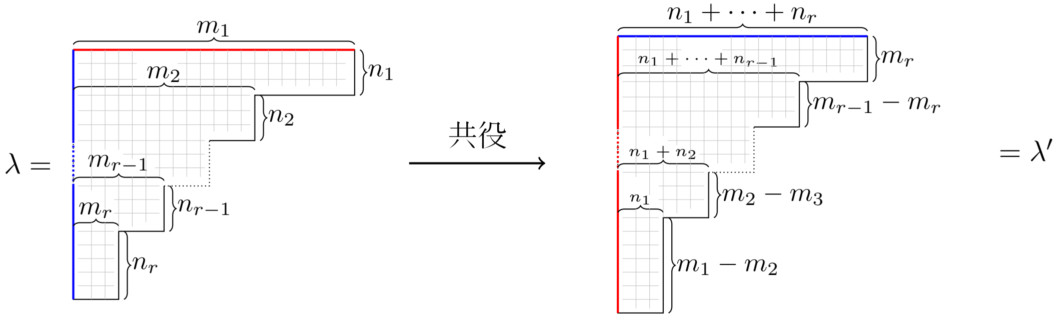
$\mathrm{Part}_r(N)$: Partitions of N made out of r different parts.

Any element $\lambda \in \mathrm{Part}_r(N)$ can be represented by a Young diagram

$\lambda =$



On $\mathrm{Part}_r(N)$ we have an involution given by the conjugation $\lambda \mapsto \lambda'$ of Young diagrams.



Since we sum over all elements in $\mathrm{Part}_r(N)$ we obtain linear relations among g

$$g\binom{k_1,\dots,k_r}{d_1,\dots,d_r} = \sum_{N>0}\left(\sum_{\lambda\in\mathrm{Part}_r(N)} f(\lambda)\right)q^N = \sum_{N>0}\left(\sum_{\lambda\in\mathrm{Part}_r(N)} f(\lambda')\right)q^N = \sum * g\binom{*,\dots,*}{*,\dots,*}.$$

Example In depth one the conjugation is just the interchange $m \leftrightarrow n$ and we have for any $k \geq 1, d \geq 0$

$$g\binom{k}{d} = \sum_{\substack{m>0\\n>0}} \frac{m^d n^{k-1}}{(k-1)!} q^{mn} = \frac{d!}{(k-1)!} \sum_{\substack{m>0\\n>0}} \frac{m^d n^{k-1}}{d!} q^{mn} = \frac{d!}{(k-1)!} g\binom{d+1}{k-1}.$$

In depth two the conjugation is given by the variable change $m_1 \to n_1 + n_2$ and $m_2 \to n_1$

$$\begin{aligned} g\binom{1,1}{1,2} &= \sum_{\substack{m_1>m_2>0\\n_1,n_2>0}} m_1 m_2^2 \, q^{m_1n_1+m_2n_2} = \sum_{\substack{m_1>m_2>0\\n_1,n_2>0}} (n_1+n_2) n_1^2 \, q^{m_1n_1+m_2n_2} \\ &= 6 \sum_{\substack{m_1>m_2>0\\n_1,n_2>0}} \frac{n_1^3}{6} \, q^{m_1n_1+m_2n_2} + 2 \sum_{\substack{m_1>m_2>0\\n_1,n_2>0}} \frac{n_1^2 n_2}{2} \, q^{m_1n_1+m_2n_2} \\ &= 6g\binom{4,1}{0,0} + 2g\binom{3,2}{0,0} = 6g(4,1) + 2g(3,2)\,. \end{aligned}$$

Combining the harmonic product and the conjugation gives another way to express the product.

$$
\begin{aligned}
g(2)g(3) = g\binom{2}{0} g\binom{3}{0} &= \frac{1}{2} g\binom{1}{1} g\binom{1}{2} \\
&= \frac{1}{2}\left(g\binom{1,1}{1,2} + g\binom{1,1}{2,1} + g\binom{2}{3} - g\binom{1}{3} \right) \\
&= g\binom{2,3}{0,0} + 3g\binom{3,2}{0,0} + 6g\binom{4,1}{0,0} + 3g\binom{4}{1} - 3g\binom{4}{0}.
\end{aligned}
$$

Using $q\frac{d}{dq}g\binom{3}{0} = 3g\binom{4}{1}$ we obtain

$$
g(2)g(3) = g(2,3) + 3g(3,2) + 6g(4,1) - 3g(4) + q\frac{d}{dq}g(3)\,,
$$

which looks (again as in the SZ-duality example) similar to $\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$.

Corollary

For $k_1, k_2 \geq 1$ and $k = k_1 + k_2$ we have

$$g(k_1)g(k_2) = \sum_{j=1}^{k-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) g(j, k-j) + \binom{k-2}{k_1-1} \left(q \frac{d}{dq} \frac{g(k-2)}{k-2} - g(k-1) \right) + \delta_{k_1,1}\delta_{k_2,1} g(2) \,,$$

where $\delta_{i,j} = \begin{cases} 1\,, & i = j \\ 0\,, & i \neq j \end{cases}$ denotes the Kronecker delta.

Recall the shuffle product for MZVs in lowest depths $(k_1, k_2 \geq 2)$:

$$\zeta(k_1) \cdot \zeta(k_2) = \int \dots \cdot \int \dots = \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1+k_2-j) \,.$$

Proposition (B.-Kühn, 2017)

The space $\mathcal{Z}_q$ is spanned by the double indexed g.

In particular, we can apply the relation obtained from the conjugation of partitions to any element in $\mathcal{Z}_q$.

Proposition (Brindle, 2021)

Applying the conjugation to $\zeta_q^{\mathrm{SZ}}(\mathbf{k})$ gives exactly $\zeta_q^{\mathrm{SZ}}(\mathbf{k}^\dagger)$.

Conjecture (B., 2014)

- The space $\mathcal{Z}_q$ is spanned by the single indexed g.
- All relations in $\mathcal{Z}_q$ are obtained from the conjugation and the harmonic product.

Theorem (B.-Burmester, 2023)

For $k_1, \dots, k_r \geq 1$ there exist $G(k_1, \dots, k_r) \in \mathbb{Q}[\![q]\!]$ such that

- In depth $r = 1$ they are the classical Eisenstein series $G(k) = -\frac{B_k}{2k!} + \frac{1}{(k-1)!} \sum_{m,n \geq 1} n^{k-1} q^{mn}$.
- They satisfy the harmonic product formula.
- For $k_1 \geq 2$ we have $\lim_{q \to 1} (1-q)^{k_1 + \cdots + k_r} G(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r)$.
- They (conjecturally) span the space $\mathcal{Z}_q$.
- They are (conjecturally) graded by weight.

(More generally, we construct $G\binom{k_1, \dots, k_r}{d_1, \dots, d_r}$ by using the double indexed g)

Example

$$G(2,1,1) = \frac{1}{1440} + \frac{1}{6} g(2) - g(2,1) + g(2,1,1),$$
$$\Delta \doteq 28G(9,3) + 150G(7,5) + 168G(5,7) - \frac{5197}{691} G(12)\,.$$

Let us answer the questions we had:

Questions

- What about the shuffle product? shuffle = conjugation + harmonic + conjugation
- Iterated integrals? (Medina - Ebrahimi-Fard - Manchon / Takeyama): Iterated Jackson integrals, Rota-Baxter Operators, q-Multiple Polylogarithm
- What are the relations? Conjecturally conjugation + harmonic product
- Dimension? (B.-Kühn / Okounkov) There exist analogues of the Zagier conjecture and the Broadhurst-Kreimer conjecture for $\mathcal{Z}_q$
- Good choice of polynomials P_i? Depends: for conjugation: ζ_q^{SZ}, for $q\frac{d}{dq}$: g, for classical duality: ζ_q^{BZ}, "correct objects": Combinatorial MES
- Why the condition $\deg(P_j) \leq k_j$? This ensures that we can write elements as "polynomials in partitions".

- **partition of** n: $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l \geq 1$ and $n = |\lambda| := \lambda_1 + \cdots + \lambda_l$.
- $\mathcal{P}$: the set of all partitions.
- $\mathbb{Q}^{\mathcal{P}}$: Set of all functions $\mathcal{P} \to \mathbb{Q}$.

To a function $f : \mathcal{P} \to \mathbb{Q}$, we associate

- a q-series $\langle f \rangle_q \in \mathbb{Q}[\![q]\!]$
- a degree $\deg(f) \in \mathbb{R}$
- a degree limit $\mathrm{Z}^{\deg}(f) \in \mathbb{R}$

in such a way that asymptotically for real q

$$(1-q)^{\deg(f)} \langle f \rangle_q = \mathrm{Z}^{\deg}(f) + O(1-q).$$

Further, we introduce a subspace $\mathbb{P} \subset \mathbb{Q}^{\mathcal{P}}$ such that for all $f \in \mathbb{P}$ we have $\langle f \rangle_q \in \mathcal{Z}_q$.

H. Bachmann, J.-W. van Ittersum
Partitions, Multiple Zeta Values and the q-bracket
Selecta Math. **30:3** *(2024).*

Definition (Bloch-Okounkov, 2000)

For $f : \mathcal{P} \to \mathbb{Q}$ define the **q-bracket** by

$$\langle f \rangle_q \; := \; \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda)\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} \; \in \; \mathbb{Q}[\![q]\!].$$

In case $f(\lambda)$ has at most polynomial growth in $|\lambda|$, its q-bracket is holomorphic for $|q| < 1$.

Notice that the denominator is given by $\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} = \prod_{n \geq 1} \frac{1}{1-q^n} = (q;q)_\infty^{-1}$.

Example Consider the function $f(\lambda) = |\lambda|$. Then we have

$$\langle f \rangle_q = \frac{\sum_{\lambda \in \mathcal{P}} |\lambda|\, q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}} = q \frac{d}{dq} \log \Big(\prod_{n \geq 1} \frac{1}{1-q^n} \Big) = \sum_{m,n \geq 1} m q^{mn} = \sum_{n \geq 1} \sigma_1(n) q^n = g(2).$$

Definition

The algebra of **shifted symmetric functions** is $\Lambda^* = \mathbb{Q}[Q_2, Q_3, \ldots]$, where $Q_k : \mathcal{P} \to \mathbb{Q}$ is given by

$$Q_k(\lambda) \;:=\; \beta_k + \frac{1}{(k-1)!} \sum_{i=1}^{\infty} \left((\lambda_i - i + \tfrac{1}{2})^{k-1} - (-i + \tfrac{1}{2})^{k-1} \right),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\beta_k = \left(\frac{1}{2^{k-1}} - 1\right) \frac{B_k}{k!}$ with B_k the k-th Bernoulli number.

Theorem (Bloch-Okounkov (2000))

For any $f \in \Lambda^*$ the q-series $\langle f \rangle_q$ is a quasimodular form.

$r_m(\lambda)$: the number of times m occurs as a part in the partition λ.

Definition (B.-van-Ittersum, 2024)

The space of **polynomial functions on partitions** $\mathbb{P}$ is the image of

$$\Psi : \bigoplus_{n\geq 0} \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n] \to \mathbb{Q}^{\mathcal{P}},$$

where Ψ maps the polynomial $p(x_1, \ldots, x_n, y_1, \ldots, y_n)$ to

$$\lambda \mapsto \sum_{m_1 > \ldots > m_n > 0} \sum_{r_1=1}^{r_{m_1}(\lambda)} \cdots \sum_{r_n=1}^{r_{m_n}(\lambda)} p(m_1, \ldots, m_n, r_1, \ldots, r_n).$$

Example The function $f = \Psi(x_1)$ is given by

$$f(\lambda) = \sum_{m>0} \sum_{r=1}^{r_m(\lambda)} m = \sum_{m>0} r_m(\lambda) m = |\lambda|.$$

$r_m(\lambda)$: the number of times m occurs as a part in the partition λ.

Definition (B.-van-Ittersum, 2024)

The space of **polynomial functions on partitions** $\mathbb{P}$ is the image of

$$\Psi : \bigoplus_{n\geq 0} \mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n] \to \mathbb{Q}^{\mathcal{P}},$$

where Ψ maps the polynomial $p(x_1,\ldots,x_n,y_1,\ldots,y_n)$ to

$$\lambda \mapsto \sum_{m_1>\ldots>m_n>0} \sum_{r_1=1}^{r_{m_1}(\lambda)} \cdots \sum_{r_n=1}^{r_{m_n}(\lambda)} p(m_1,\ldots,m_n,r_1,\ldots,r_n).$$

Theorem (B.-van-Ittersum (2024))

- We have $\Lambda^* \subsetneq \mathbb{M} := \{f \in \mathbb{P} \mid \langle f\rangle_q \text{ is quasi-modular}\} \subsetneq \mathbb{P}$.
- For any $f \in \mathbb{P}$ we have $\langle f\rangle_q \in \mathcal{Z}_q$. (and any element in $\mathcal{Z}_q$ arises in this way)

Definition

For $\mathcal{F} = \{f_k\}_{k=1}^{\infty}$ with $f_k \in \mathbb{Q}[x]$ and $k_i \geq 1$, $d_i \geq 0$ define the following element in $\mathbb{P}$

$$P_{\mathcal{F}}\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} : \mathcal{P} \longrightarrow \mathbb{Q}$$

$$\lambda \longmapsto \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} m_j^{d_j} f_{k_j}(r_{m_j}(\lambda)).$$

Example For $\mathrm{s} = \{f_k\}_{k=1}^{\infty}$, with $f_k(x) - f_k(x-1) = \frac{1}{(k-1)!} x^{k-1}$ and $f_k(0) = 0$ we get

$$\left\langle P_{\mathrm{s}}\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} \right\rangle_q = \sum_{\substack{m_1 > \ldots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \prod_{j=1}^{r} m_j^{d_j} \frac{n_j^{k_j - 1}}{(k_j - 1)!} q^{m_j n_j} = g\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix}.$$

Definition

For $f \in \mathbb{Q}^{\mathcal{P}}$ we define the

- **degree** by

$$\deg(f) = \inf_{a \in \mathbb{R}} \left\{ \lim_{q \to 1} (1-q)^a \langle f \rangle_q \text{ converges} \right\}.$$

- **degree limit** $Z^{\deg}(f) \in \mathbb{R} \cup \{\pm\infty\}$ by

$$\lim_{q \to 1} (1-q)^{\deg(f)} \langle f \rangle_q$$

  whenever it exists

Example Consider the function $f(\lambda) = |\lambda|$. Then we have $\deg(f) = 2$ and

$$Z^{\deg}(f) = \lim_{q \to 1} (1-q)^2 \langle f \rangle_q = \lim_{q \to 1} (1-q)^2 g(2) = \zeta(2).$$

Theorem (B.-van-Ittersum, 2024)

Given $r \geq 1$ and $d_i, l_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, \ldots, r$, let $f = \Psi(\prod_{i=1}^r x_i^{d_i} y_i^{l_i}) \in \mathcal{P}$. Then,

$$\deg(f) = \max_{j \in \{0,\ldots,r\}} \left\{ \sum_{i \leq j} (d_i + 1) + \sum_{i > j} (l_i + 1) \right\}.$$

Moreover, if the maximum is attained for a unique value of j, then $\mathrm{Z}^{\deg}(f) \in \mathcal{Z}_{\leq \deg(f)}$.

Corollary

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ and $d_1, \ldots, d_{s-1} \geq 0, d_s \geq 1$ we have

$$\lim_{q \to 1} (1-q)^{k_1 + \cdots + k_r + d_1 + \cdots + d_s} g\binom{1, \ldots, 1, k_1, \ldots, k_r}{d_1, \ldots, d_s, 0, \ldots, 0} = \xi(d_1, \ldots, d_s)\zeta(k_1, \ldots, k_r),$$

where we call $\xi(d_1, \ldots, d_s)$ **conjugated multiple zeta values**.

Definition (B.-van-Ittersum, 2024)

For $d_1, \ldots, d_{r-1} \geq 0, d_r \geq 1$, define the **conjugated multiple zeta value** by

$$\xi(d_1, \ldots, d_r) \ := \sum_{0<m_1<\ldots<m_r} \frac{1}{m_1 \cdots m_r} \, \Omega\Big[\prod_{i=1}^{r} \Big(\frac{1}{m_i} + \ldots + \frac{1}{m_r}\Big)^{d_i}\Big],$$

where $\Omega : \mathbb{Q}[m_1^{-1}, \ldots, m_r^{-1}] \to \mathbb{Q}[m_1^{-1}, \ldots, m_r^{-1}]$ is the linear mapping

$$\Omega\Big[\frac{1}{m_1^{l_1} \cdots m_r^{l_r}}\Big] \ := \ \frac{l_1! \cdots l_r!}{m_1^{l_1} \cdots m_r^{l_r}} \, .$$

These satisfy the index-shuffle product formula, e.g.,

$$\xi(d_1)\xi(d_2) = \xi(d_1, d_2) + \xi(d_2, d_1)$$

for $d_1, d_2 \geq 1$.

Definition (van-Ittersum, 2021)

The vector space isomorphism $\langle\ \rangle_{\vec{u}} : \mathbb{Q}^{\mathcal{P}} \to \mathbb{Q}[\![u_1, u_2, \ldots]\!]$ is given by

$$\langle f \rangle_{\vec{u}} := \frac{\sum_{\lambda\in\mathcal{P}} f(\lambda)\, u_\lambda}{\sum_{\lambda\in\mathcal{P}} u_\lambda} \qquad (u_\lambda = u_{\lambda_1} u_{\lambda_2} \cdots,\ u_0 = 1).$$

For $f \in \mathbb{Q}^{\mathcal{P}}$ we call $\langle f \rangle_{\vec{u}}$ the **$\vec{u}$-bracket** of f.

Note that the $\vec{u}$-bracket reduces to the q-bracket by specializing $u_i = q^i$ for all integers i.

Definition

Given $F, G \in \mathbb{Q}[\![u_1, u_2, \ldots]\!]$, we define

- the **harmonic product** as the multiplication $F \circledast G = FG$.
- the **conjugation** of $F = \sum_{\lambda \in \mathcal{P}} a_\lambda u_\lambda$ by $\iota(F) = \sum_{\lambda \in \mathcal{P}} a_\lambda u_{\lambda'}$.
- the **shuffle product** as the multiplication $F \sqcup\!\sqcup G = \iota(\iota(F) \circledast \iota(G))$.
- the **derivative** of $F = \sum_{\lambda \in \mathcal{P}} a_\lambda u_\lambda$ by $DF = \sum_{\lambda \in \mathcal{P}} a_\lambda |\lambda|\, u_\lambda$.

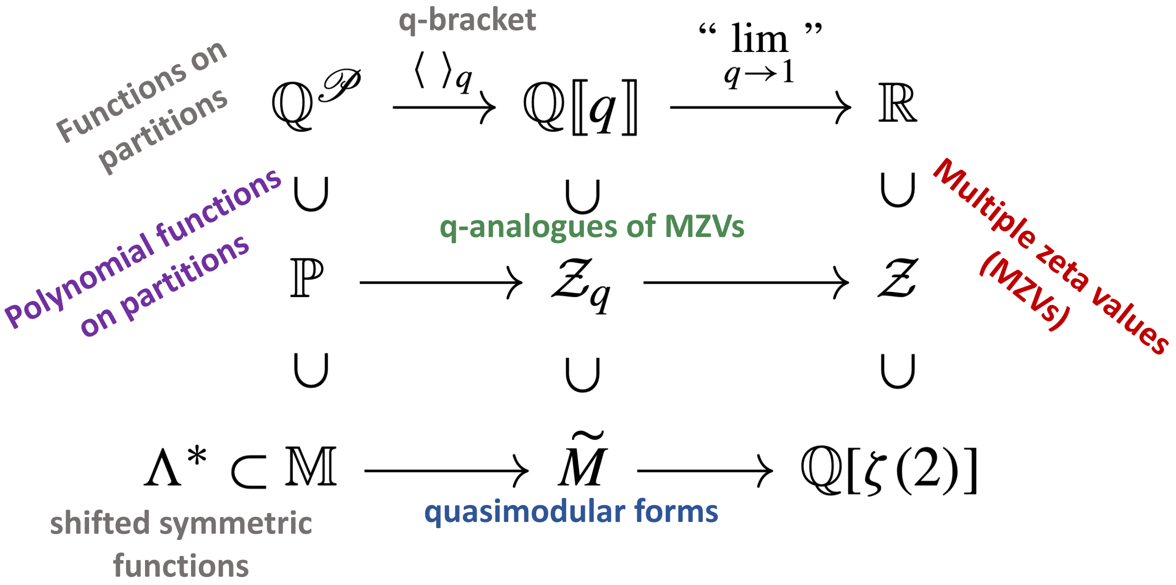
We extend these definitions to $\mathbb{Q}^{\mathcal{P}}$ by the isomorphism given by the $\vec{u}$-bracket.

Proposition (Double shuffle relations for general functions on partitions)

For all $f, g \in \mathbb{Q}^{\mathcal{P}}$ we have

$$\langle \iota(f) \rangle_q = \langle f \rangle_q\,, \quad \langle f \circledast g \rangle_q = \langle f \rangle_q \langle g \rangle_q = \langle f \sqcup\!\sqcup g \rangle_q \quad \text{and} \quad q\frac{\partial}{\partial q}\langle f \rangle_q = \langle Df \rangle_q\,.$$

Question: Applications to other (modular) objects than q-analogues?



Functions on partitions
q-bracket
$\langle\ \rangle_q$
“ $\lim_{q\to 1}$ ”
$\mathbb{Q}^{\mathscr{P}}$
$\mathbb{Q}[[q]]$
$\mathbb{R}$
Polynomial functions on partitions
q-analogues of MZVs
Multiple zeta values (MZVs)
$\mathbb{P}$
$\mathcal{Z}_q$
$\mathcal{Z}$
$\Lambda^* \subset \mathbb{M}$
$\widetilde{M}$
$\mathbb{Q}[\zeta(2)]$
shifted symmetric functions
quasimodular forms