1 2 3 π e $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Numbers, infinite sums and multiple zeta values

17^{th} YLC Seminar - 25th December 2018

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Goal of this talk

1. Talk about the classification of numbers

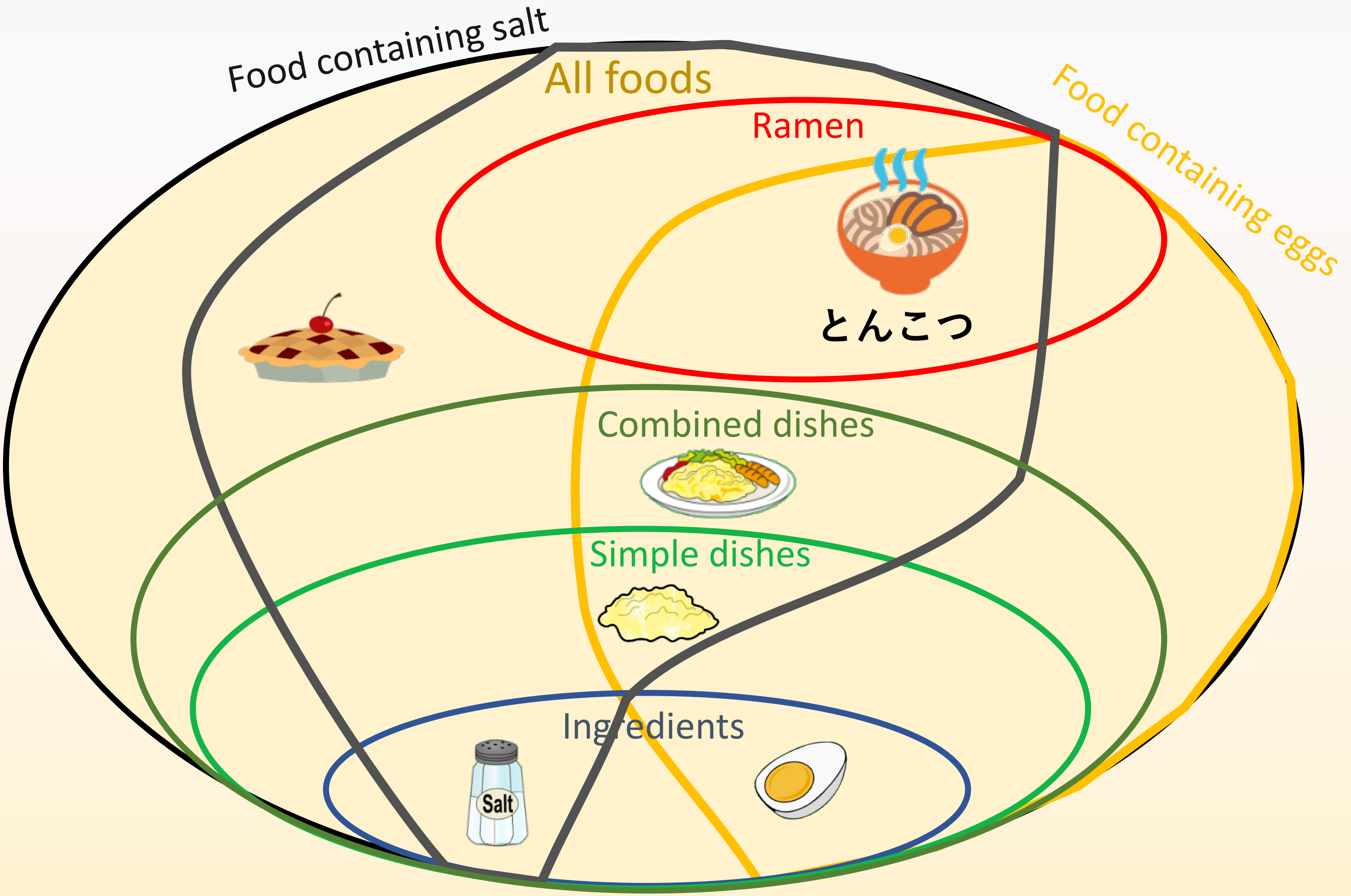
   What does that mean?

2. Infinite sums

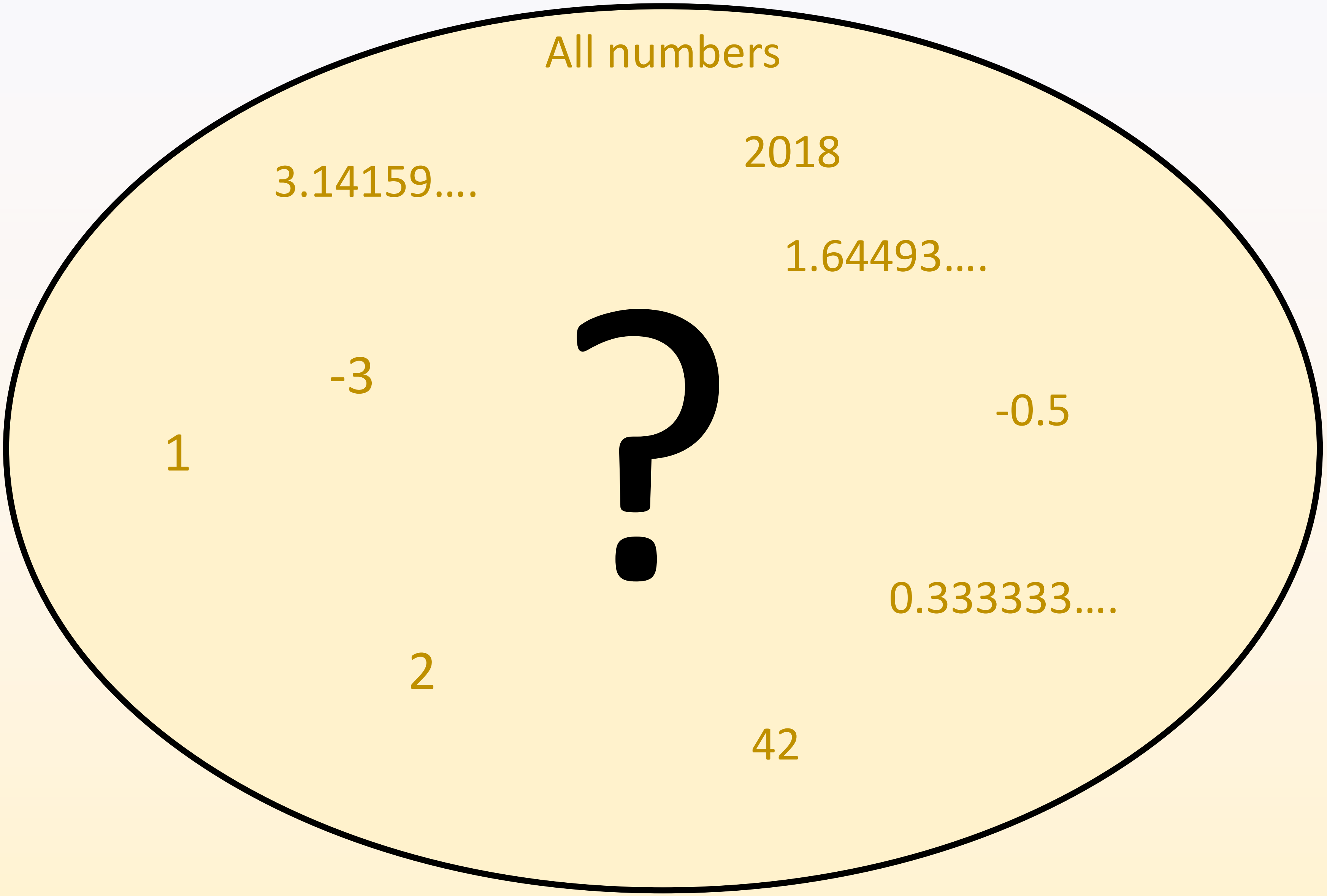
3. Multiple zeta values

4. Sum of divisors and q-analogues

Classification of food



Classification of numbers

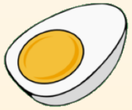


The numbers 1,2,3,4,…. are called natural numbers

We can add (+) and multiply (∗) natural numbers

$$6 = 2 * 3$$

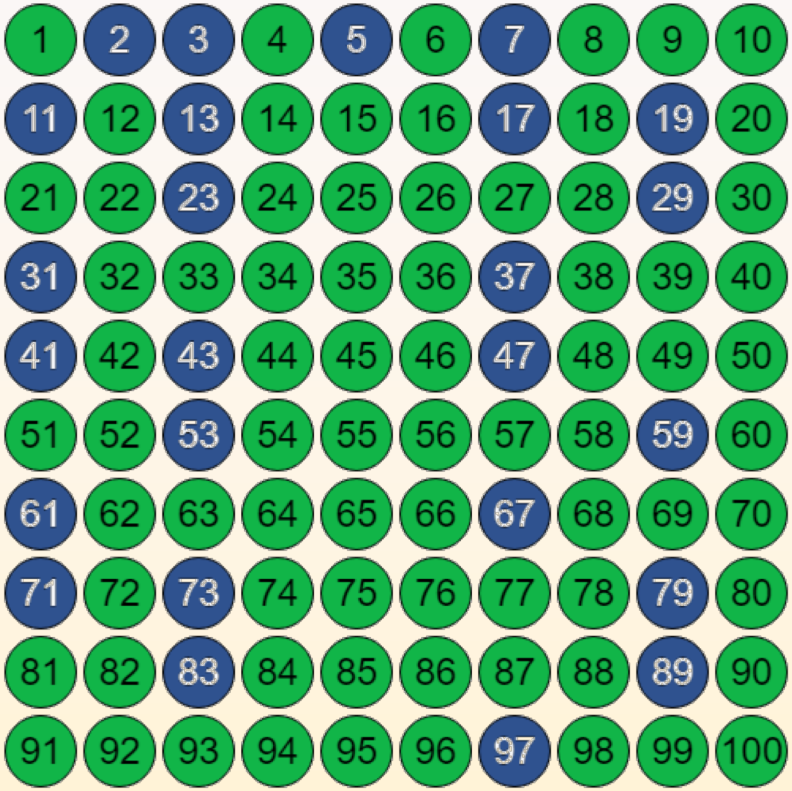
 = 

∗ 

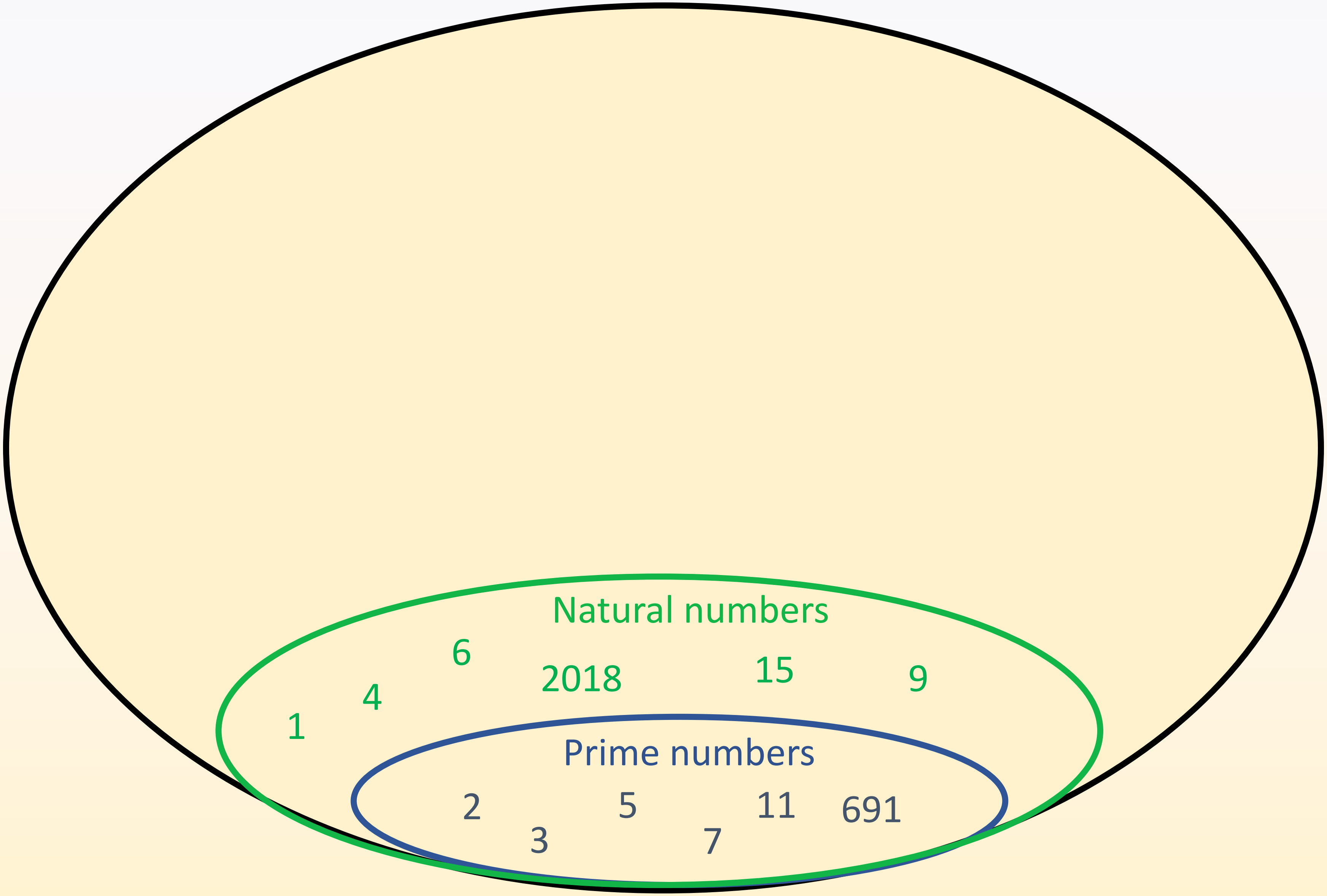
What are the ingredients for natural numbers?

Prime numbers the ingredients…

A natural number greater than 1 is called a **prime number**, if it can not be written as a product of two smaller numbers.



Classification of numbers … so far



Integers

combined dishes...

We also have zero 0 and negative numbers -1, -2, -3, -4,...

The natural numbers together with 0 and their negatives are called **integers**.

<u>Mathematicians point of view</u>

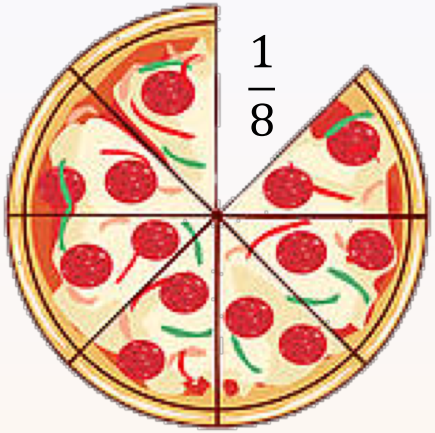
A, B : natural numbers

Integers allow us to solve the following equation for X:

$$X + B = A$$

For example $X = -5$ is the solution of $X + 7 = 2$.

Rational numbers combined dishes...



Numbers given by fractions are called **rational numbers**.

$$\frac{a}{b}$$

a ← numerator

b ← denominator

Mathematicians point of view

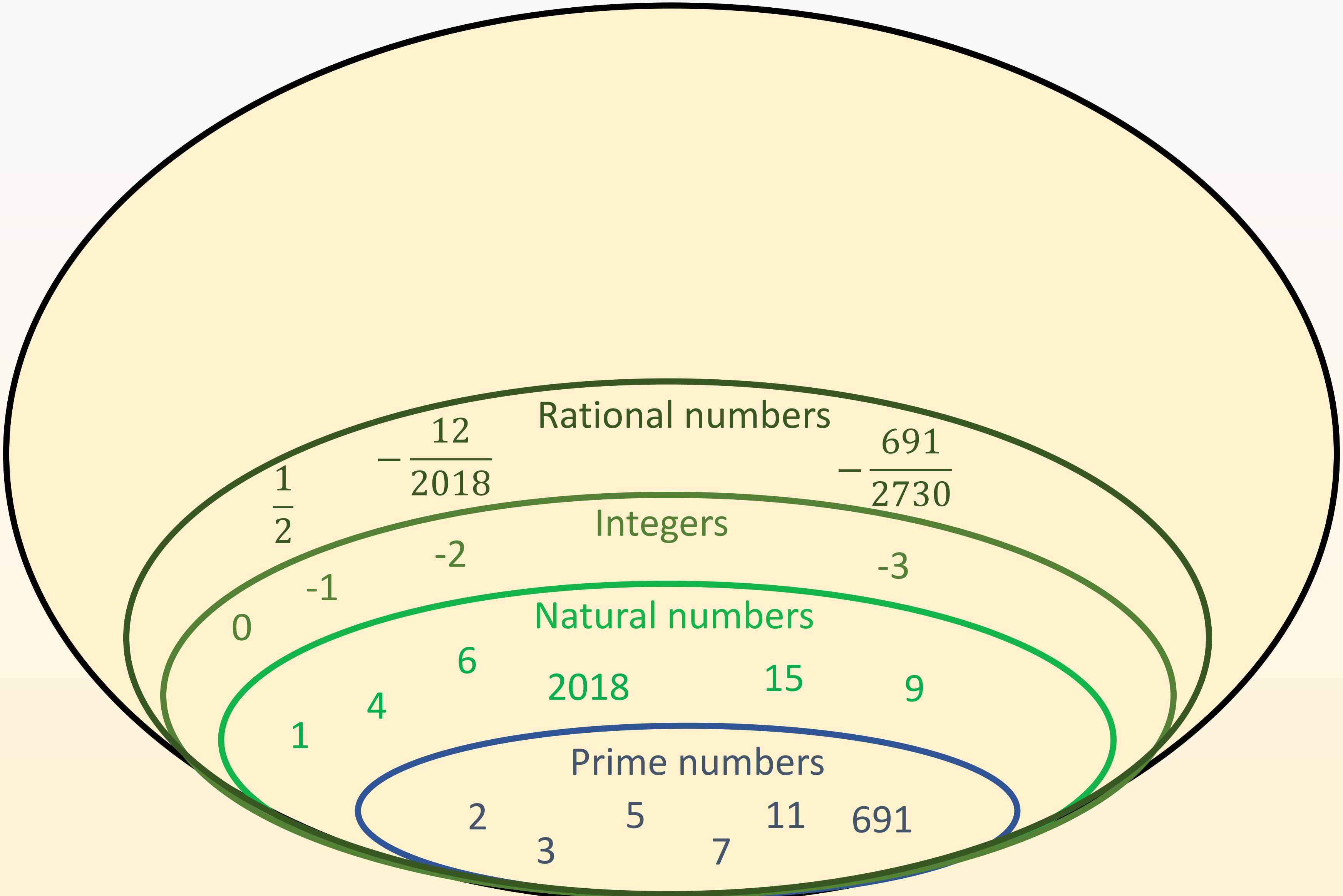
A, B, C : natural numbers

Rational numbers allow us to solve the following equation for X:

$$C\,X + B = A$$

For example $X = \frac{2}{5}$ is the solution of $5\,X + 7 = 9$.

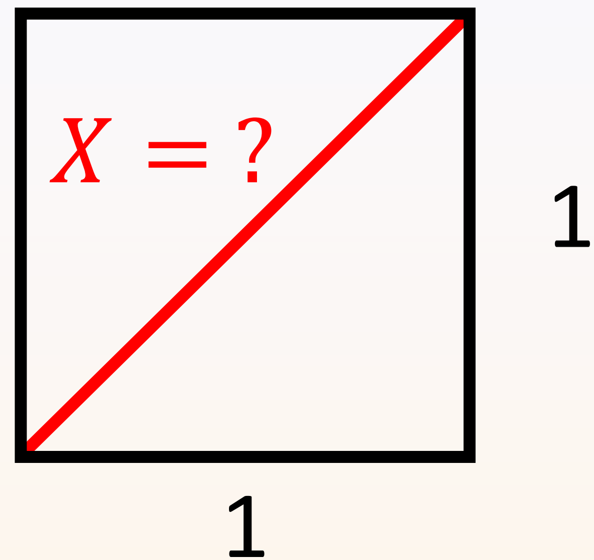
Classification of numbers … so far



Algebraic numbers

$$X^2 = 1^2 + 1^2 = 2$$

$$X = \sqrt{2} \approx 1.414\ldots$$



<u>Mathematicians point of view</u>

$A_0, A_1, \ldots, A_n$: integers

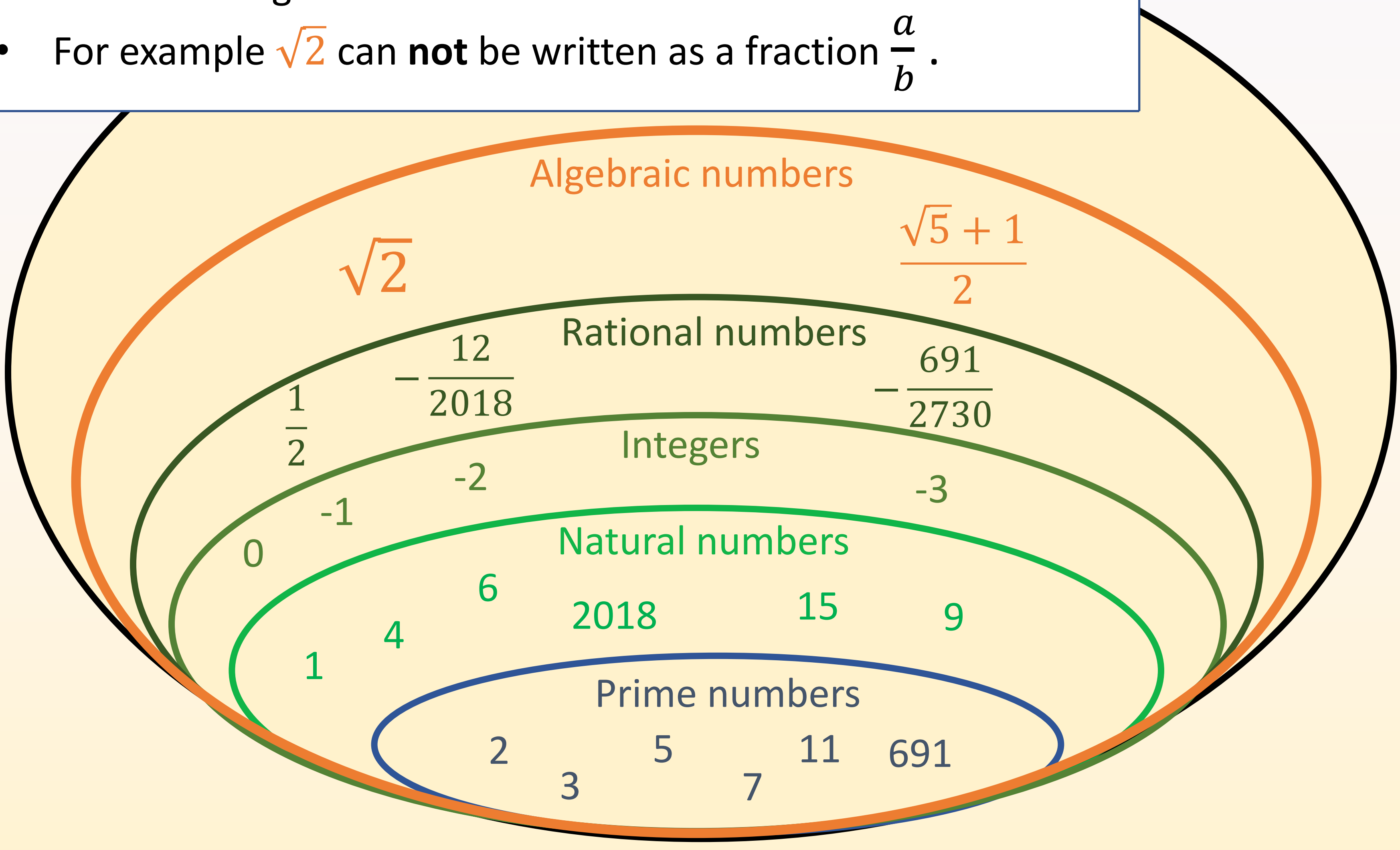
Algebraic numbers are given as solutions for X of polynomial equations:

$$A_n X^n + \ldots + A_2 X^2 + A_1 X + A_0 = 0$$

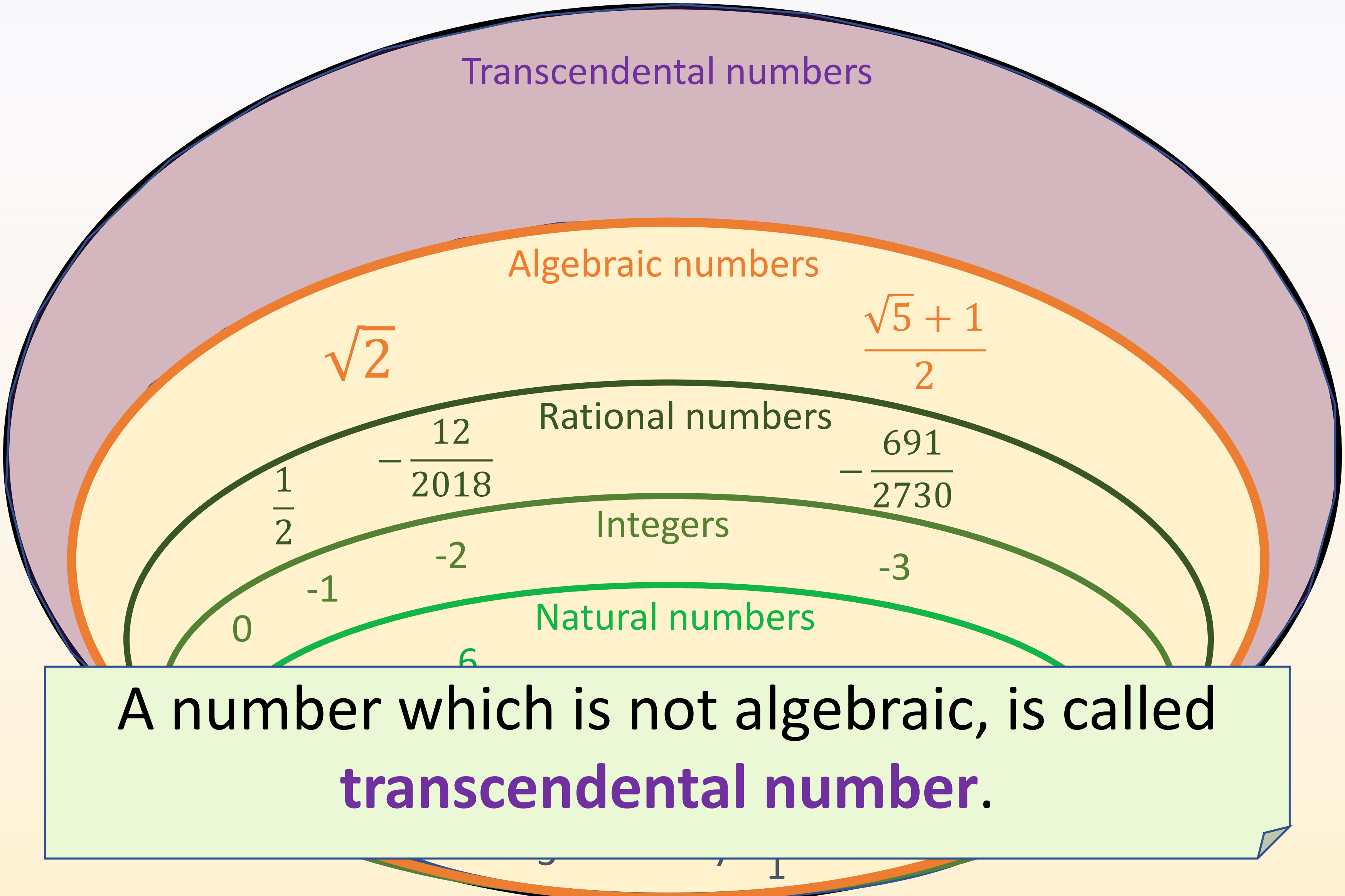
For example $X = \sqrt{2}$ is the solution of $1\,X^2 + 0\,X - 2 = 0$.

Classification of numbers … so far

- Every rational number is also an algebraic number.
- But not all algebraic numbers are rational!
- For example $\sqrt{2}$ can **not** be written as a fraction $\frac{a}{b}$.

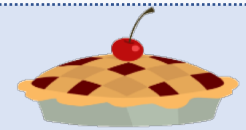


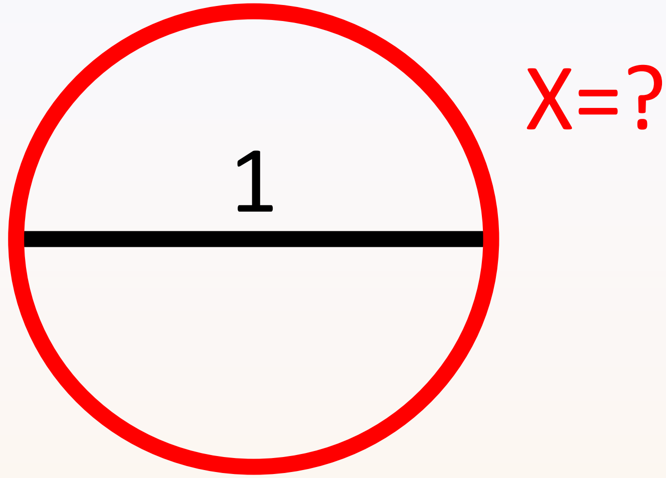
Transcendental numbers



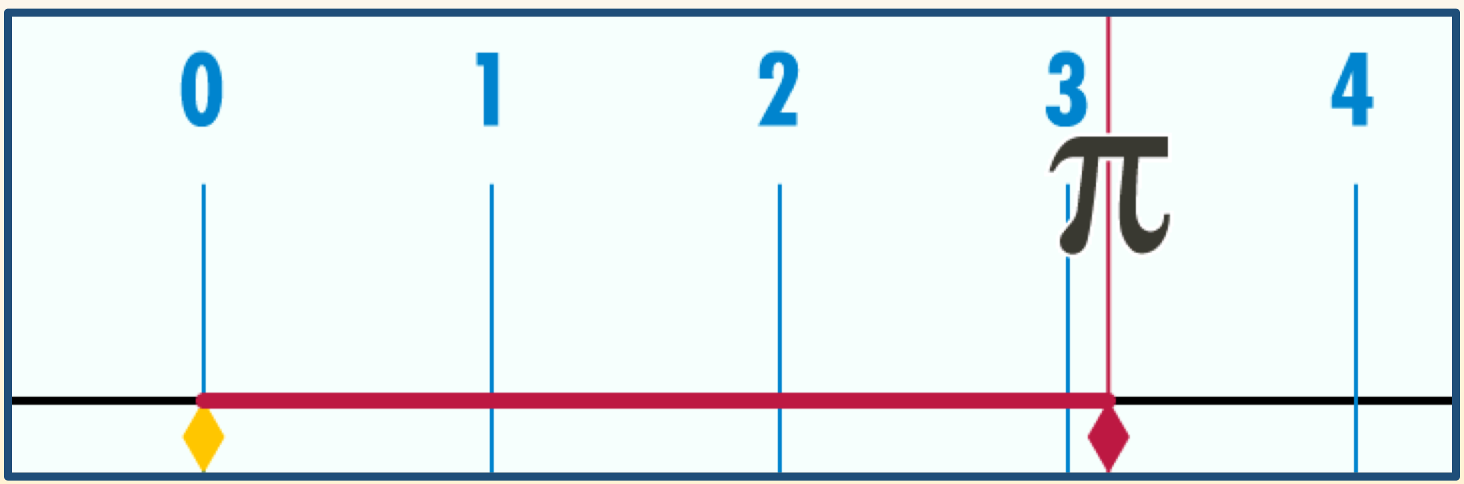
A number which is not algebraic, is called **transcendental number**.

π

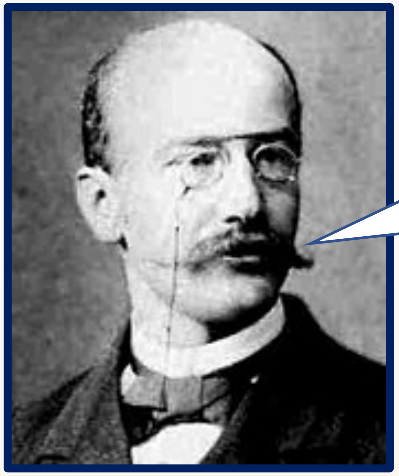




$$\pi = 3.141592.....$$



Pi π



Ferdinand von Lindemann
(1852 – 1939)

π is transcendental!

This means you will **never** find integers $A_0, A_1, \ldots, A_n$ such that

$$A_n \pi^n + \ldots + A_2\pi^2 + A_1\pi + A_0 = 0$$

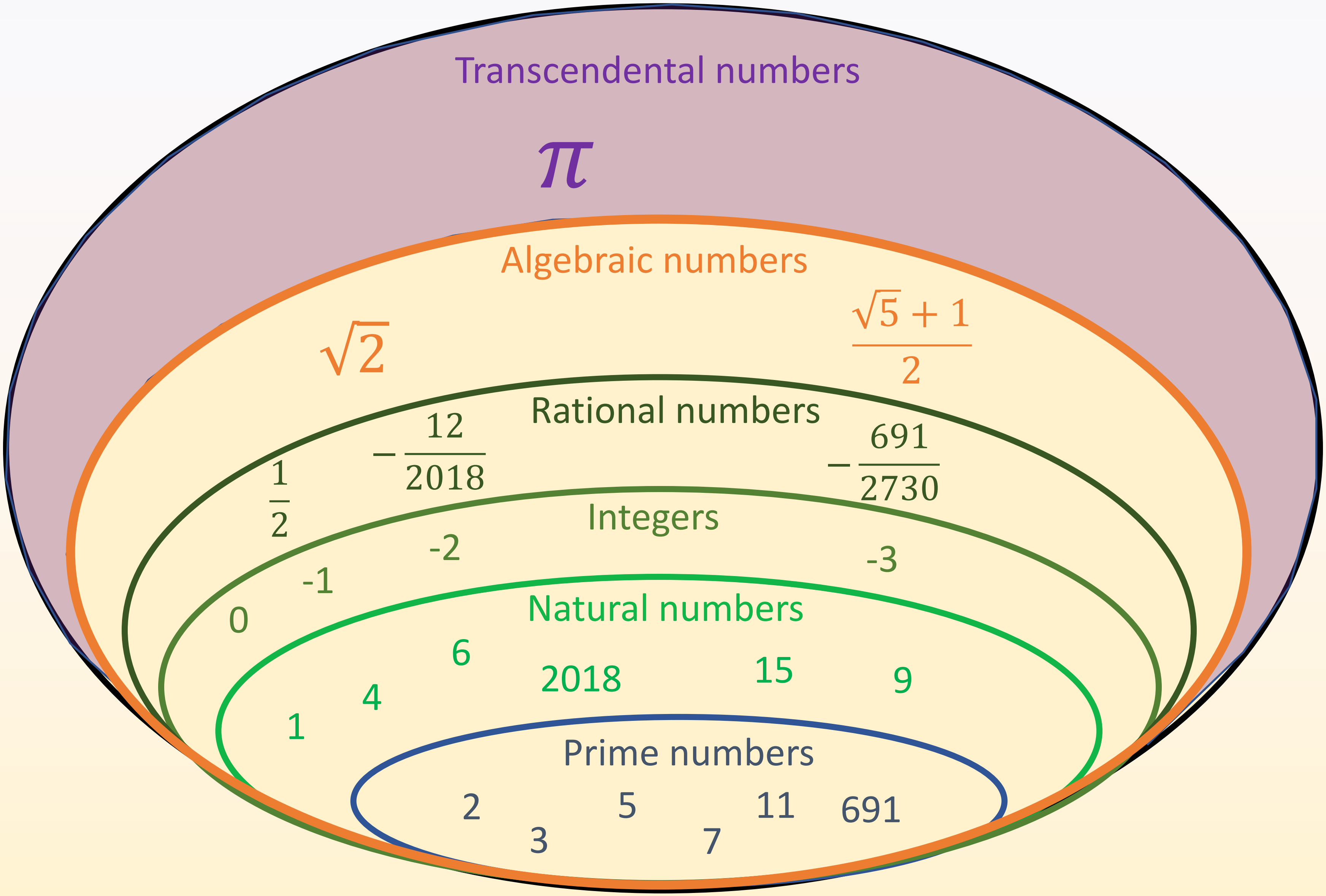
$$7\pi - 22 \neq 0$$

$$2\pi^2 - 6\pi - 1 \neq 0$$

$$\pi^3 - 22\pi^2 + 4\pi - 12 \neq 0$$

....

Classification of numbers … so far



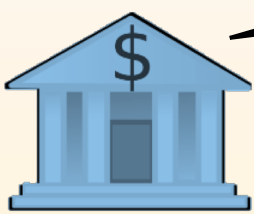
Nice banks

We offer you an interest rate of 100% each year!

Nice bank

After one year: $(1+1)=2$

We also offer you an interest rate of 100% each year.
But at our bank you get 50% after every 6 month!



Nicer bank

After 6 month: $\left(1+\frac{1}{2}\right)=1.5$

After one year: $\left(1+\frac{1}{2}\right)*\left(1+\frac{1}{2}\right)=2.25$

Nice banks

Nice bank

After one year: $(1+1)=2$

Nicer bank

After one year: $\left(1+\frac{1}{2}\right)^{2}=2.25$

We apply the interest after every month!

Super nice bank

After one year: $\left(1+\frac{1}{12}\right)^{12}=2.613\ldots$

We apply the interest after every day!

Super super nice bank

After one year: $\left(1+\frac{1}{365}\right)^{365}=2.714\ldots$

The nicest bank & Euler's number

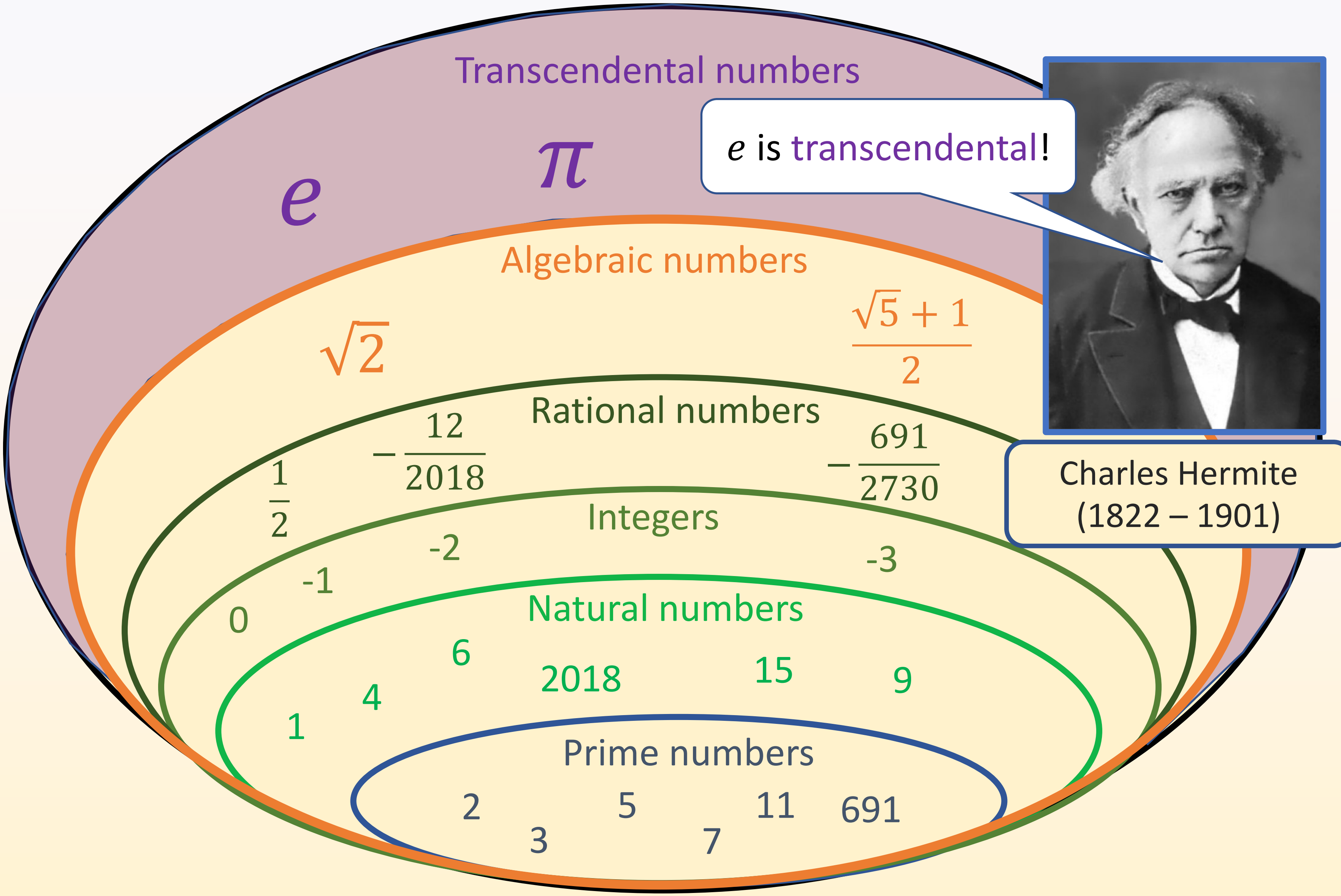
The nicest bank applies the interest rate continuously ("infinitely often")

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845\ldots$$

The number e is called **Euler's number**.

Leonhard Euler
(1707 – 1783)

Classification of numbers … so far



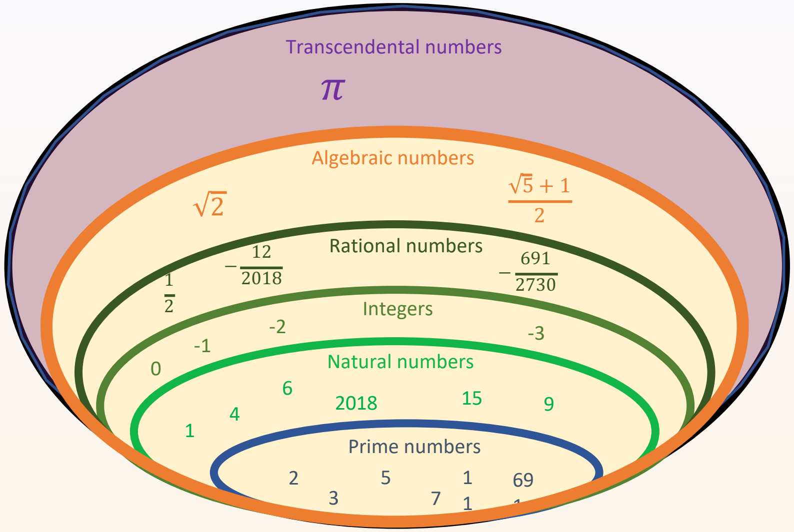
Quiz

Question 1
Suppose c is rational and A and B are algebraic.

i) Is $c * A$ algebraic?

ii) Is $A * B$ algebraic?

iii) Is A+B algebraic?



Question 2
Suppose c is rational and not zero and A and B are transcendental.

i) Is $c * A$ transcendental?

ii) Is $A * B$ transcendental? $$\pi * \frac{1}{\pi} = 1$$

iii) Is A+B transcendental? $$\pi + (-\pi) = 0$$

Infinite sums

Finite sums:

$$1 = 1$$

$$1 + 2 = 3$$

$$1 + 2 + 3 = 6$$

$$1 + 2 + 3 + 4 = 10$$

…

$$1 + 2 + 3 + \ldots + 100 = 5050$$

This sum gets bigger and bigger and therefore the infinite sum

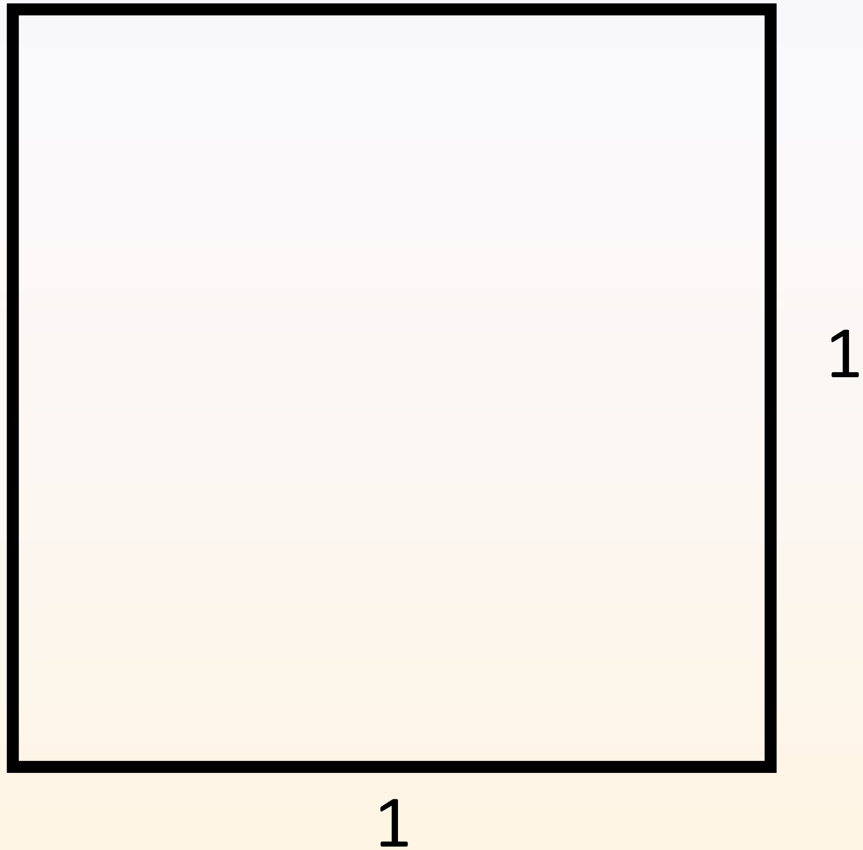
$$\xcancel{1 + 2 + 3 + 4 + \ldots + 100 + \ldots + 43432423} + \ldots\ldots..$$

does not make sense.

But there are infinite sums which make sense!

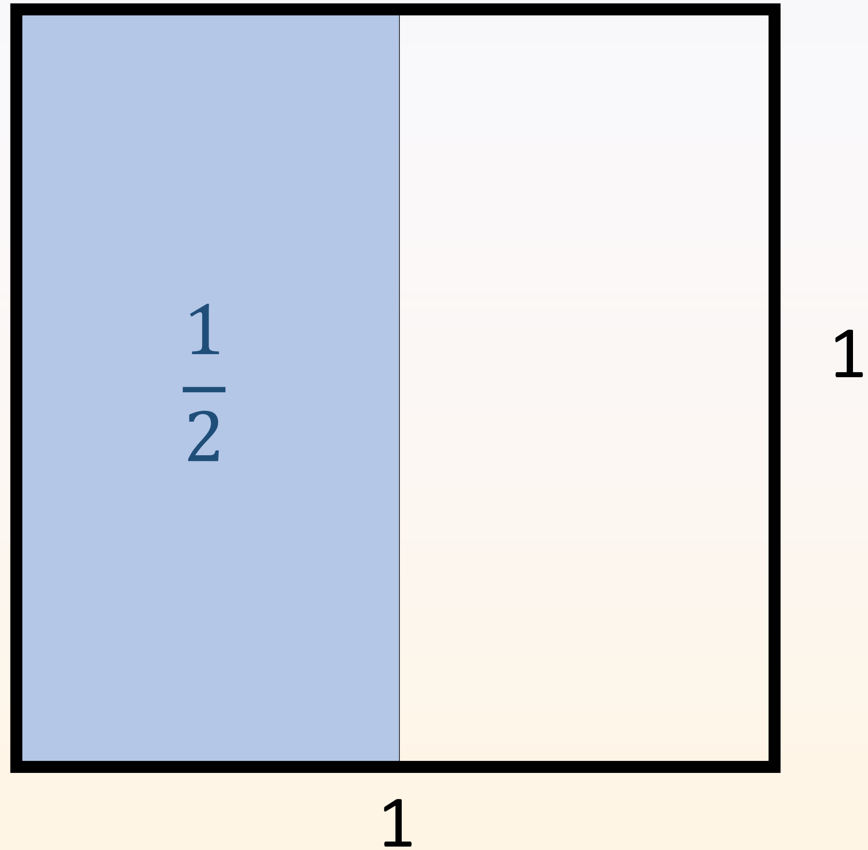
$$1 + 0.1 + 0.01 + 0.001 + 0.0001 + \ldots = 1.1111111\ldots$$

Infinite sums - Example



Area of $\square$ = 1

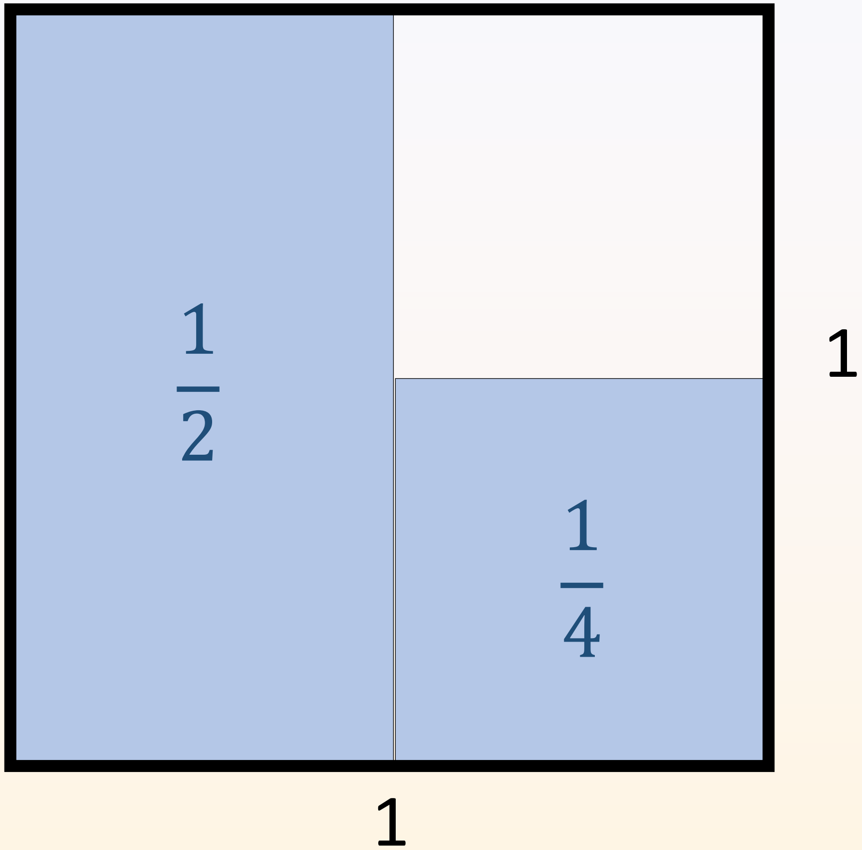
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2} = 0.5$

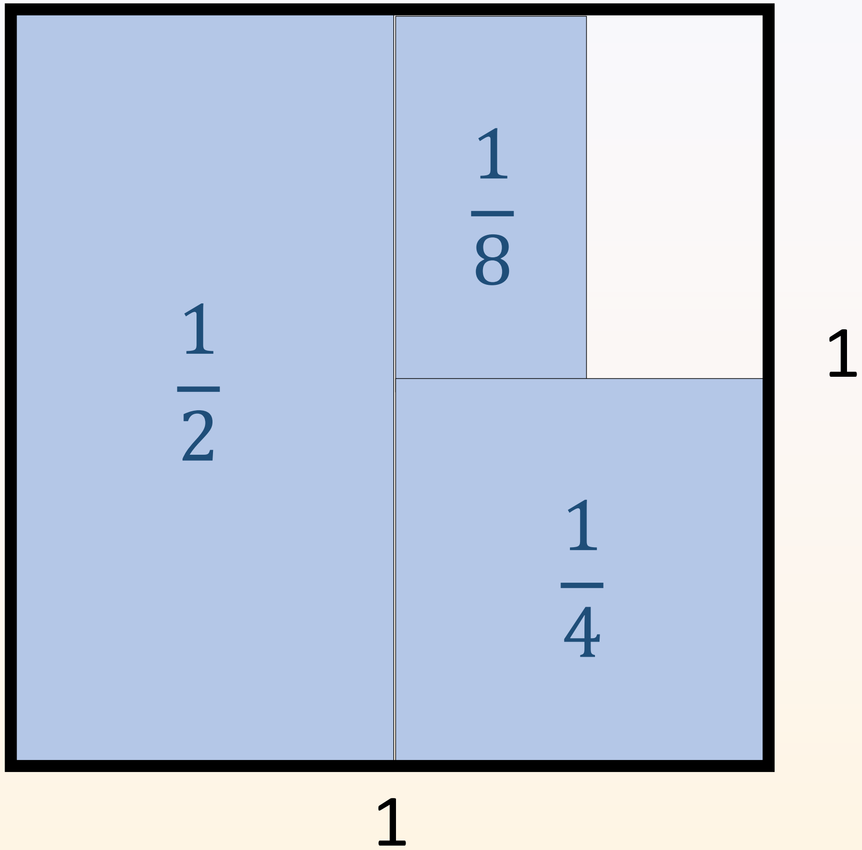
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2} + \frac{1}{4} = 0.75$

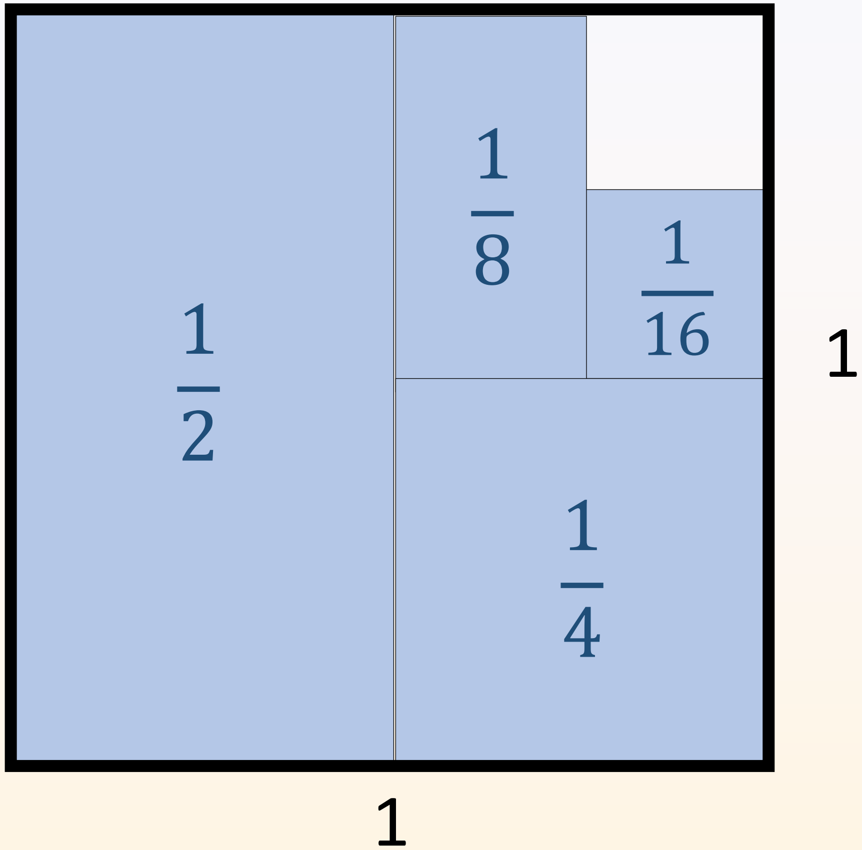
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=0.875$

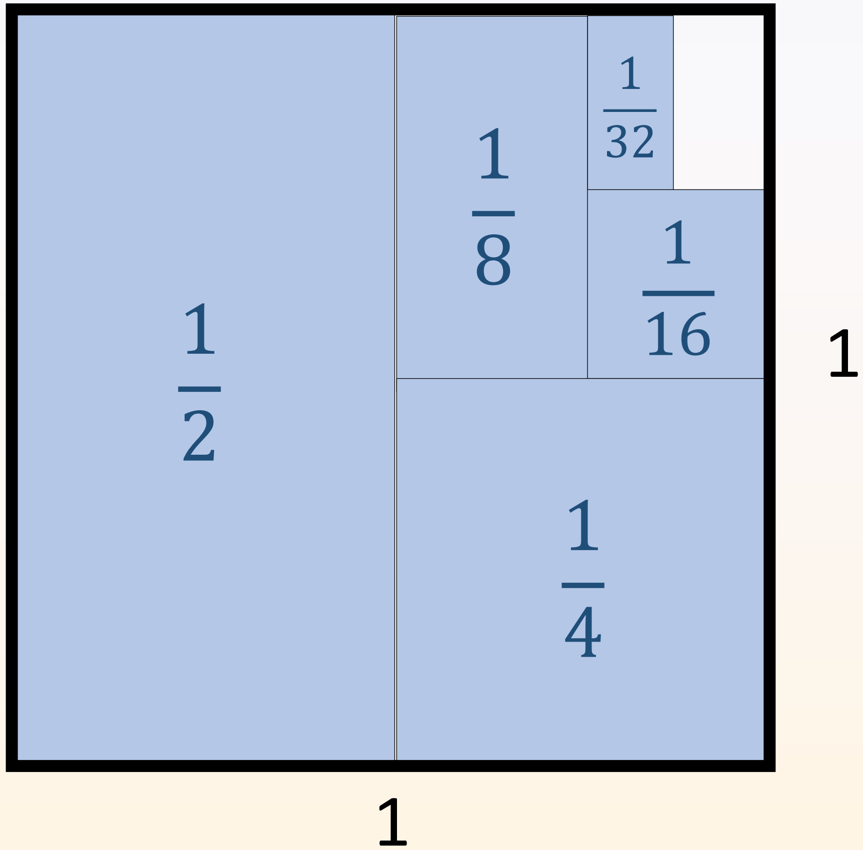
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = 0.93\ldots$

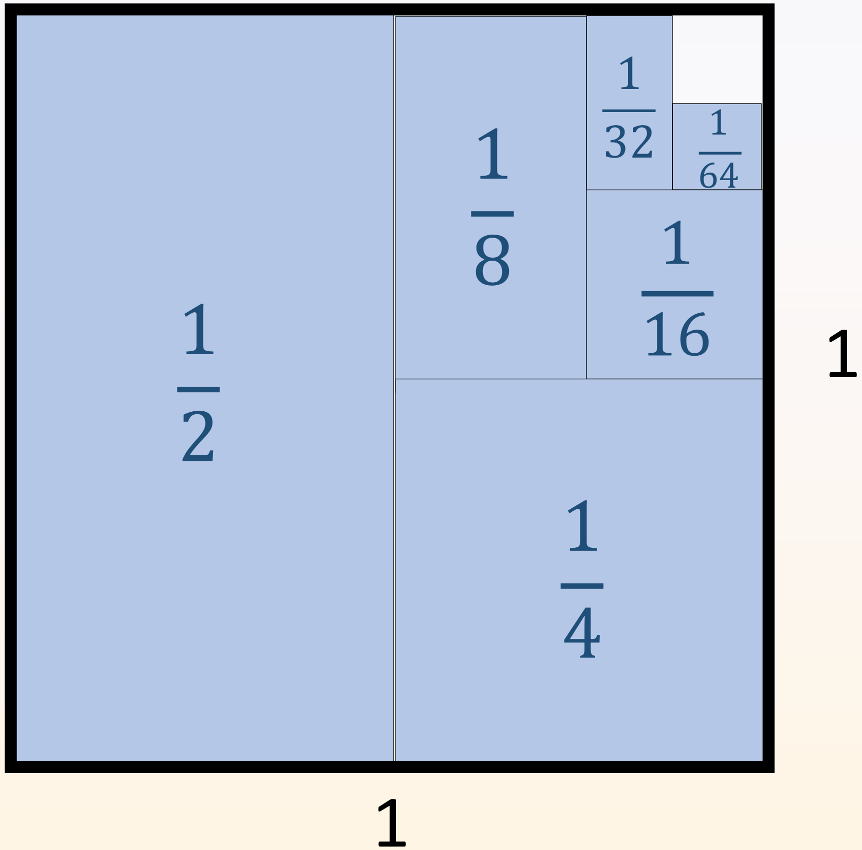
Infinite sums - Example



Area of $\square$ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=0.96\ldots$

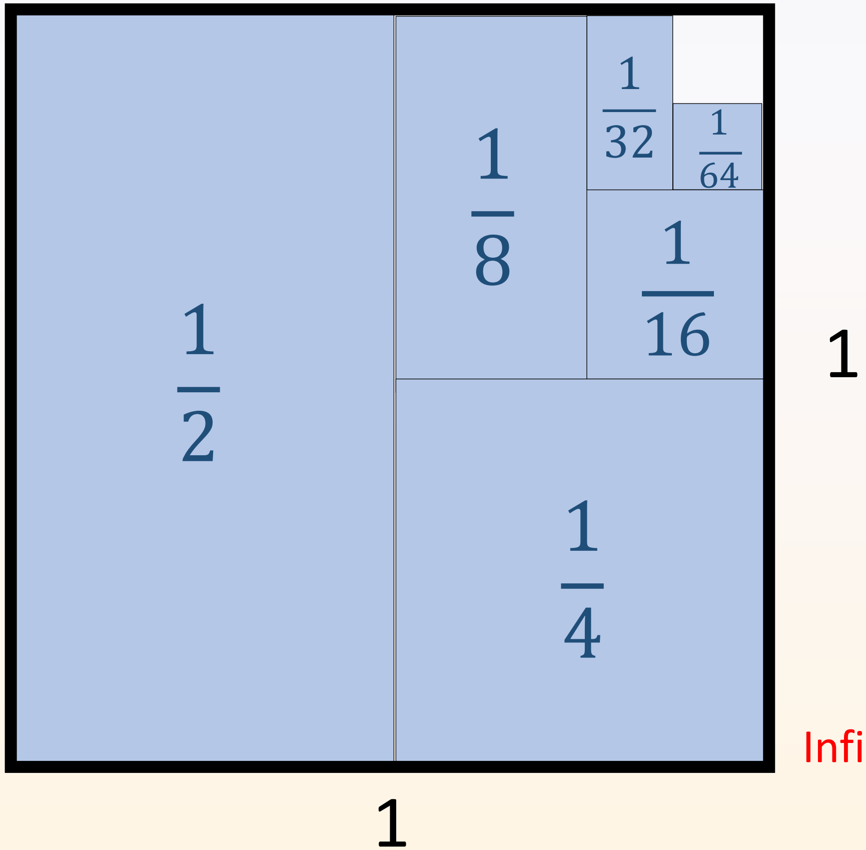
Infinite sums - Example



Area of $\square$ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}=0.98\ldots$

Infinite sums - Example



Infinite number of terms (no end)

Area of $\square$ $= 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$

Area of blue part $= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.98\ldots$

Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$

Mathematical notation:

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots=1$$

For any natural number A greater than 1 we have

$$\sum_{n=1}^{\infty}\frac{1}{A^n}=\frac{1}{A-1}$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = 1$$

What happens if we change this sum a little bit?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = ??$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = ???$$

$$\frac{1}{1^2} = 1$$

$$\frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = 1.25$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = 1 + \frac{1}{4} + \frac{1}{9} = 1.3611\ldots$$

$$\frac{1}{1^2} + \cdots + \frac{1}{100^2} = 1 + \cdots + \frac{1}{10000} = 1.6349\ldots$$

$$???$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

But where is the circle??

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}$$

Leonhard Euler
(1707 – 1783)

Riemann zeta values

For any natural number k greater than 1 the numbers

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

are called **Riemann zeta values.**

Euler's formulas imply:

If k is even then $\zeta(k)$ is transcendental

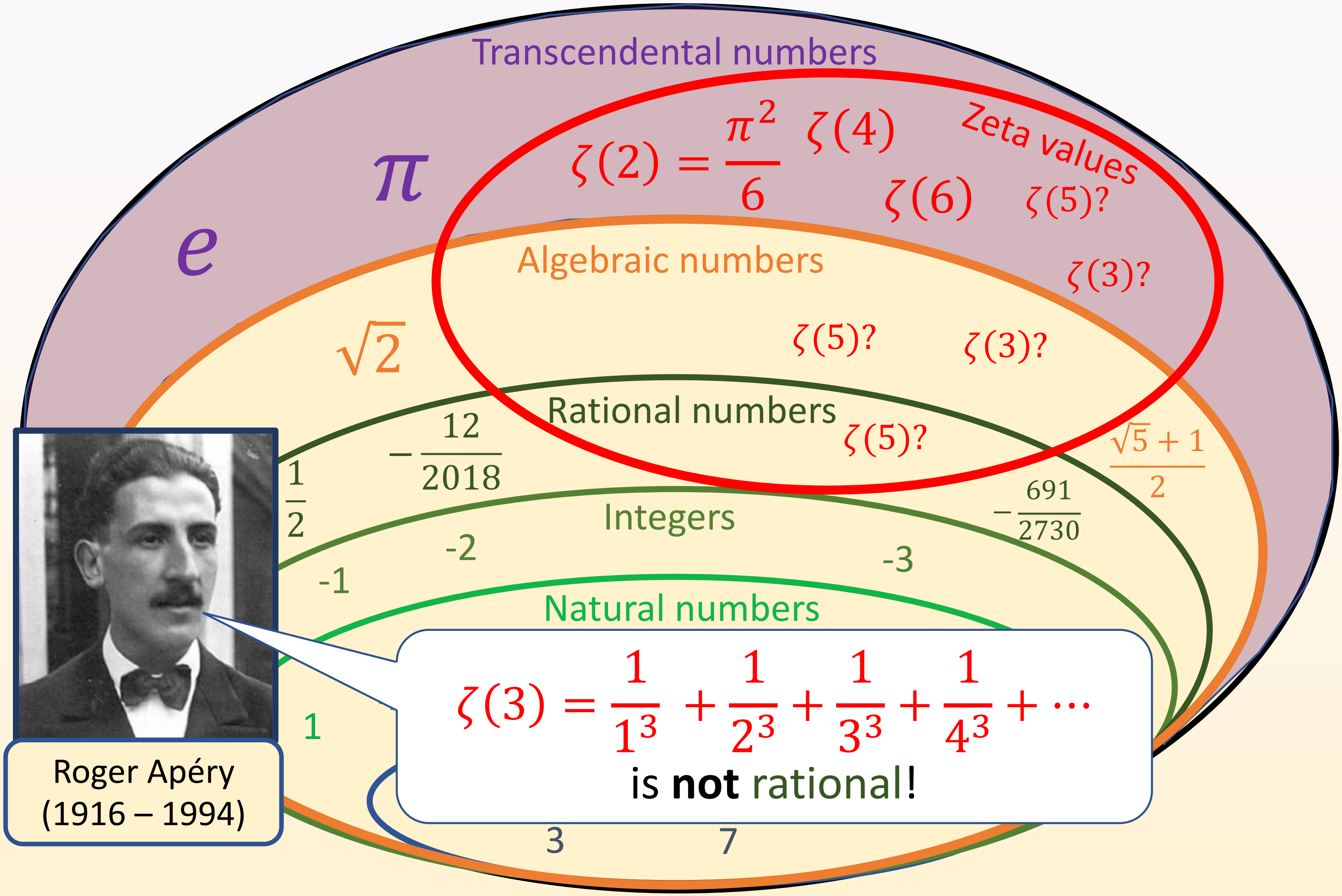
Conjecture:

$\zeta(k)$ is transcendental for all natural numbers k greater than 1

Bernhard Riemann (1826 – 1866)

Pictures: wikipedia.org

Classification of numbers



Multiple zeta values the ramen…

The **Riemann zeta values** can also be written as:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \sum_{n>0} \frac{1}{n^k} = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots$$

Sum over all integers n, which satisfy the condition $n > 0$

The **double zeta values** are defined by

$$\zeta(r,s) = \sum_{0<m<n} \frac{1}{m^r n^s} = \frac{1}{1^r 2^s} + \frac{1}{1^r 3^s} + \frac{1}{2^r 3^s} + \frac{1}{1^r 4^s} + \cdots$$

Sum over all integers m and n, which satisfy the condition $0 < m < n$

Multiple zeta values the ramen…

For $k_1, k_2, \ldots, k_{r-1} \geq 1$ and $k_r \geq 2$ the **multiple zeta values** are defined by

$$\zeta(k_1, \ldots, k_r) = \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

These numbers satisfy a lot of relations

Examples:

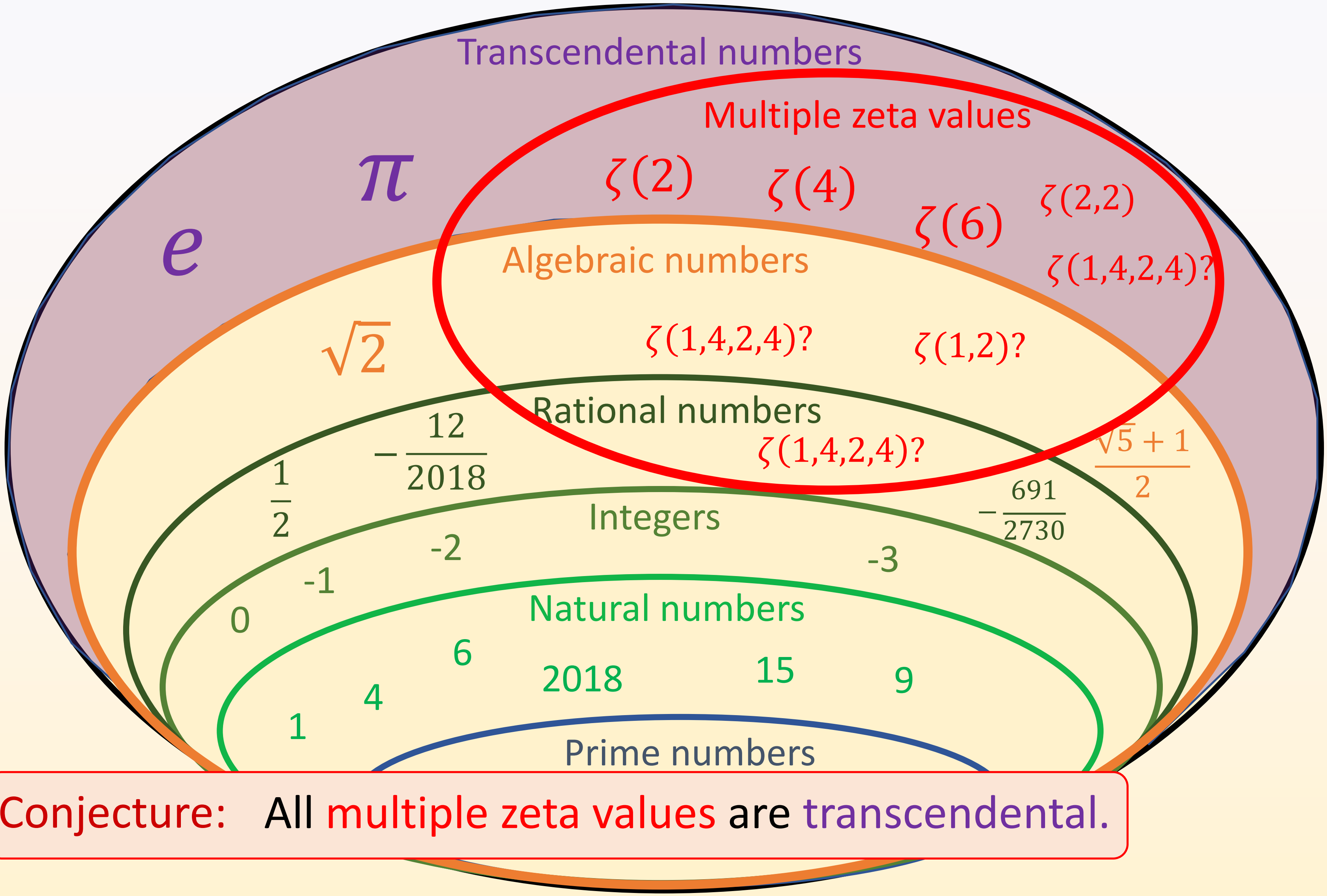
$$\zeta(3) = \zeta(1,2)$$

$$\frac{5197}{691}\zeta(12) = 168\zeta(7,5) + 150\zeta(5,7) + 28\zeta(3,9)$$

$$\zeta(\underbrace{2, \ldots, 2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}$$

One of the goals is to understand all these relations

Classification of numbers



Conjecture: All multiple zeta values are transcendental.

Multiple zeta values the ramen…

$$\zeta(k_1,\ldots,k_r) = \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}$$

The product of two multiple zeta values is again a linear combination of multiple zeta values

$$\zeta(r)\zeta(s) = \sum_{m>0}\frac{1}{m^r}\sum_{n>0}\frac{1}{n^s} = \sum_{\substack{m>0\\n>0}}\frac{1}{m^r n^s}$$

$$= \sum_{0<m<n}\frac{1}{m^r n^s} + \sum_{0<n<m}\frac{1}{m^r n^s} + \sum_{0<m=n}\frac{1}{m^{r+s}}$$

$$= \zeta(r,s) + \zeta(s,r) + \zeta(r+s)$$

Sum of divisors

For a natural number n, the **sum of divisor function** is defined by

$$\sigma(n) = \sum_{d|n} d$$

Sum over all integers **d** which divide **n**

Examples:

$$\sigma(6) = \sum_{d|6} d = 1 + 2 + 3 + 6 = 12$$

n	1	2	3	4	5	**6**	7	8
$\sigma(n)$	1	3	4	7	6	**12**	8	15

Sum of divisors

n	1	2	3	4	5	6	7	8
$\sigma(n)$	1	3	4	7	6	12	8	15

We put these numbers into an infinite sum with a parameter q:
(called a **q-series**)

$$S(q) = \sum_{n>0} \sigma(n) q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + \dots$$

The $S(q)$ is a function in q and it makes sense for $0 < q < 1$.

$$S(0.5) = 2.7\ldots, \quad S(0.8) = 30.8\ldots, \quad S(0.9) = 143.4\ldots, \quad S(0.99) = 16262.9\ldots$$

$$\lim_{q \to 1} S(q) = \infty$$

Sum of divisors what happens near q=1?

$$S(q) = \sum_{n>0} \sigma(n) q^n = q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + \ldots$$

$$\lim_{q \to 1} S(q) = \infty \qquad \lim_{q \to 1} (1-q) = 0$$

$S(q)$ VS $1-q$

$$\lim_{q \to 1} (1-q) S(q) = \infty$$

$$\lim_{q \to 1} (1-q)^2 S(q) = \zeta(2)$$

$$\lim_{q \to 1} (1-q)^3 S(q) = 0$$

q-analogues

> *In mathematics, a **q-analogue** of a theorem, identity or expression is a generalization involving a new parameter q that returns the original theorem, identity or expression in the limit as q → 1.*
>
> Wikipedia

$$\lim_{q\to 1} \overbrace{(1-q)^2 S(q)}^{\textbf{q-analogue} \text{ of } \zeta(2)} = \zeta(2)$$

Modular form

*In **my research** I am interested in properties of various different models of q-analogues of multiple zeta values and their connections to modular forms.*

Thank you very much and

Merry Christmath